An Exact Sequence of Koszul Transforms of Modules

Michinori Sakaguchi

(Received on February 20, 1998)

In commutative ring theory many interesting facts about regular sequences are obtained. Schmitt developed in [2] the theory of regular sequences of $\mathbb{Z}_2$-graded ring, in which he introduced the concept of odd regular sequnces. Let $R = R_0 + R_1$ be $\mathbb{Z}_2$-graded ring and let M be an graded R-module. If r is an even homogeneous element in R, we mean that r is an M-regular element in classical sense. On the other hand Schmitt defined an odd M-regular element $\rho \in R_1$ by satisfying the condition that the sequence

$$A \xrightarrow{\rho} A \xrightarrow{\rho} A$$

is exact. It seems quite different to the classical one, but many corepponding results are shown in [2].

If R is a neotherian local ring and M is a finite and non-zero R-module, then we can consider the maximal length of odd M-sequence in the same way as the defition of depth M. We call it *odd depth* of M and denote it by odepth M. He got a fundamental relation

$$\text{odepth } M = \text{odepth } M/\rho M + 1$$

for an M-regular element $\rho \in R_1$ between odepths as we have one between depths. However its proof needs several steps and the theory of cones in projective varieties. In this paper we shall show an exact sequence of Koszul transforms (Theorem 1) which enables us to show another proof of the above equation.

As for $\mathbb{Z}_2$-graded ring we reffer to Chapter 3 in [1].

In this paper we fix notation as follows. Let R be a $\mathbb{Z}_2$-graded noetherian local ring with residue field k and let $\mathfrak{m}$ be the maximal ideal of R. Let A be a finite, non-zero and left R-module and $\rho_1 \in R_1$. We put $\bar{R} = R/\rho_1 R$ and $\bar{A} = A/\rho_1 A$. Assume that ρ_1 is regular on A. Then it follows from Theorem 2.6 in [1] that $\bar{\rho}_1 \neq 0$ in Φ, where $\Phi = (m/m^2)_1$. Therefore if $\dim_k \Phi \geq 2$, we can choose elements $\rho_2, \ldots, \rho_q$ in R_1 such that $\{\bar{\rho}_1, \bar{\rho}_2, \ldots, \bar{\rho}_q\}$ is a basis of Φ over k. Let $r_1, \ldots, r_p$ be elements in $\mathfrak{m}_0$ such that $\{\bar{r}_1, \ldots, \bar{r}_p\}$ is a basis of $(\mathfrak{m}/\mathfrak{m}^2)_0$ over k. Let $\xi_1, \ldots, \xi_p$ be odd variable over R and let $x, y_2, \ldots, y_q$ be even variables over R. We denote the polynomial ring $R[\xi_1, \ldots, \xi_p \mid x, y_2, \ldots, y_q]$ by $R[x \mid x, y]$ and denote $R[\xi_1, \ldots, x_p \mid y_2, \ldots, y_q]$ by $R[\xi \mid y]$. We consider odd elements

$$\delta = \sum r_i\xi_i + \rho_1 x + \sum_{i\geq 2}\rho_i y_i \in R[\xi \mid x, y]$$

$$\delta' = \sum r_i\xi_i + \sum_{i\geq 2}\rho_i y_i \in R[\xi \mid y]$$

$$\bar{\delta} = \sum \bar{r}_i\xi_i + \sum_{i\geq 2}\bar{\rho}_i y_i \in \bar{R}[\xi \mid x, y]$$

where $\bar{r}_i$ and $\bar{\rho}_i$ are respectively the image of r_i and ρ_i under the canonical homomorphism $R \longrightarrow \bar{R}$. The *Koszul transform* $\mathfrak{C}_A$ is the homology of a complex

$$A[\xi \mid x, y] \xrightarrow{\ \delta\ } A[\xi \mid x, y] \xrightarrow{\ \delta\ } A[\xi \mid x, y]$$

where $A[\xi \mid x, y] = A \otimes_R R[\xi \mid x, y]$. $\mathfrak{C}_A$ will be also denoted by $\mathfrak{C}(\delta, A[\xi \mid x, y])$. By Lemma 4.1 in [2] $\mathfrak{C}_A$ is also an $k[\xi \mid x, y]$-module. Let $z \in A[\xi \mid x, y]$ with $\delta z = 0$. We denote the class of z in $\mathfrak{C}_A$ by $[z]$. We put $\mathfrak{C}_{\bar{A}} = \mathfrak{C}(\delta, \bar{A}[\xi \mid x, y])$, $\mathfrak{C}'_{\bar{A}} = \mathfrak{C}(\delta', \bar{A}[\xi \mid y])$ and $\bar{\mathfrak{C}}_{\bar{A}} = \mathfrak{C}(\bar{\delta}, \bar{A}[\xi \mid y])$ since $\bar{A}$ is also an $\bar{R}$-module.

Lemma 1. *$\bar{\mathfrak{C}}_{\bar{A}}$ is naturally isomorphic to $\mathfrak{C}'_{\bar{A}}$ and we have $\mathfrak{C}_{\bar{A}} = \mathfrak{C}'_{\bar{A}}[x]$.*

Proof. Let $\bar{z}$ be an elements in $\bar{A}[\xi \mid y]$. We see that

$$\bar{\delta}\bar{z} = \delta\bar{z} = (\delta' + \rho_1 x)\bar{z} = \delta'\bar{z}.$$

This implies that $\bar{\mathfrak{C}}_{\bar{A}}$ is naturally isomorphic to $\mathfrak{C}'_{\bar{A}}$. Let $\bar{v}$ be an element in $\bar{A}[\xi \mid x, y]$. Simce $\bar{A}[\xi \mid x, y] = \bar{A}[\xi \mid y][x]$, we can express an element $\bar{v}$ of $\bar{A}[\xi \mid x, y]$ with $\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_l \in \bar{A}[\xi \mid y]$ such that

$$\bar{v} = \bar{z}_0 + \bar{z}_1 x + \cdots + \bar{z}_l x^l.$$

It follows from the fact $\rho_1\bar{A} = 0$ that

$$\delta\bar{v} = (\delta' + \rho_1 x)\bar{v} = \delta'\bar{z}_0 + \delta'\bar{z}_1 x + \cdots + \delta'\bar{z}_l x^l.$$

Hence $\delta\bar{v} = 0$ if and only if $\delta'\bar{z}_i = 0$ for all i. Therefore we can define a mapping $\phi : \mathfrak{C}_{\bar{A}} \rightarrow \mathfrak{C}'_{\bar{A}}[x]$ by $\phi([\bar{v}]) = [\bar{z}_0] + [\bar{z}_1]x + \cdots + [\bar{z}_l]x^l$. It is easily seen that ϕ is an isomorhism in the category of $R[\xi \mid x, y]$-modules and we may write as $\mathfrak{C}_{\bar{A}} = \mathfrak{C}'_{\bar{A}}[x]$. This proves our lemma.

Because ρ_1 is regular on A, we get a short exact sequence

$$0 \longrightarrow \bar{A} \xrightarrow{\ \rho_1\ } A \longrightarrow \bar{A} \longrightarrow 0.$$

Since $R[\xi \mid x, y]$ is an R-flat module,

$$0 \longrightarrow \bar{A}[\xi \mid x, y] \xrightarrow{\ \rho_1\ } A[\xi \mid x, y] \longrightarrow \bar{A}[\xi \mid x, y] \longrightarrow 0$$

is exact, which makes a complex exact sequence. Whence we obtain a long exact squence

$$\longrightarrow \mathfrak{C}_{\bar{A}} \xrightarrow{\Delta} \mathfrak{C}_{\bar{A}} \xrightarrow{\rho_1} \mathfrak{C}_A \longrightarrow \mathfrak{C}_{\bar{A}} \xrightarrow{\Delta} \mathfrak{C}_A \longrightarrow,$$

where Δ is the connecting homomorphism.

Lemma 2. *Let v be an element of $A[\xi \mid x, y]$ with $\delta v = 0$. Then there exists an element z in $A[\xi \mid y]$ such that $\delta z = 0$ and $[z] = [v]$ in $\mathfrak{C}_A$.*

Proof. As the proof of Lemma 1, we can write

$$v = z_0 + z_1 x + \cdots + z_l x^l$$

with $z_i \in A[\xi \mid y]$ for all i. We prove by induction on l. If $l = 0$, then it is trivial. Suppose $l \geq 1$. Put

$$v_1 = z_0 + z_1 x + \cdots + z_{l-1} x^{l-1}.$$

Then $v = v_1 + z_l x^l$. Comparing the coefficient of x^{l+1} of the equation

$$\delta v = (\delta' + \rho_1 x)(v_1 + z_l x^l) = 0,$$

one gets $\rho_1 z_l = 0$. Hence there exists $w_l \in A[\xi \mid y]$ such that $z_l = \rho_1 w_l$, since ρ_1 is regular on A. Therefore

$$\begin{aligned} v &= v_1 + \rho_1 w_l x^l = v_1 + \rho_1 x w_l x^{l-1} \\ &= v_1 + (\delta - \delta') w_l x^{l-1} = (v_1 - \delta' w_l x^{l-1}) + \delta w_l x^{l-1}. \end{aligned}$$

By induction we can find $z \in A[\xi \mid y]$ and $v' \in A[\xi \mid x, y]$ such that $\delta z = 0$, $v_1 - \delta' w_l x^{l-1} = z + \delta v'$, because $\delta(v_1 - \delta' w_l x^{l-1}) = 0$. This yields

$$v = z + \delta(v' + w_l x^{l-1}),$$

whence one obtains that $[v] = [z]$ in $\mathfrak{C}_A$ which proves our lemma.

Lemma 3. *The mapping $\mathfrak{C}_{\bar{A}} \xrightarrow{\rho_1} \mathfrak{C}_A$ is surjective.*

Proof. By lemma 2 each element in $\mathfrak{C}_A$ can be expressed as $[z]$ with $z \in A[\xi \mid y]$. Then $\delta z = (\delta' + \rho_1 x) z = 0$, and so $\delta' z = 0$, $\rho_1 z = 0$. Since ρ_1 is an A-regular element, we can write as $z = \rho_1 w$ with $w \in A[\xi \mid y]$. Then we see that

$$\rho_1 (\delta' w) = - \delta' \rho_1 w = - \delta' z = 0.$$

Again we can find an element w_1 in $A[\xi \mid y]$ such that $\delta' w = \rho_1 w_1$. Let $\bar{w}$ be the residue class of w in $\bar{A}[\xi \mid y]$. Then if follows that $\delta \bar{w} = \delta' \bar{w} = 0$. Therefore we obtain that $[\bar{w}] \in \mathfrak{C}_{\bar{A}}$ and $\rho_1 [\bar{w}] = [\rho_1 w] = [z]$ and this proves our lemma.

From the long exact sequence and Lemma 3, we get the following theorem.

Theorem 1. *Let A be a finite R-module and let ρ_1 be an odd and A-regular element. Then the sequence*

$$0 \longrightarrow \mathfrak{C}_{\bar{A}} \xrightarrow{\Delta} \mathfrak{C}_{\bar{A}} \xrightarrow{\rho_1} \mathfrak{C}_A \longrightarrow 0$$

is exact.

We define an $R[\xi \mid y]$-homomorphism $\phi : \mathfrak{C}'_{\bar{A}} \longrightarrow \mathfrak{C}'_{\bar{A}}$ by $\phi([\bar{z}]) = [\bar{w}]$ for $[\bar{z}] \in \mathfrak{C}'_{\bar{A}}$, where $z, w \in A[\xi \mid y]$ and $\delta' z = \rho_1 w$. Since we see by Lemma 1 that $\mathfrak{C}_{\bar{A}} = \mathfrak{C}'_{\bar{A}}[x] = \mathfrak{C}'_{\bar{A}} \otimes_{R[\xi \mid y]} R[\xi \mid x, y]$, we get an $R[\xi \mid x, y]$-homomorphism $\phi \otimes 1 : \mathfrak{C}'_{\bar{A}}[x] \longrightarrow \mathfrak{C}'_{\bar{A}}[x]$. We denote by id the identity mapping of $\mathfrak{C}_{\bar{A}}$.

Proposition 1. Under the assumption of Theorem 1, we have $\Delta = \phi \otimes 1 + x\, id$.

Proof. By Lemma 1 each element in $\mathfrak{C}_{\bar{A}}$ can be expressed as

$$[\bar{z}_0] + [\bar{z}_1]\, x + \cdots + [\bar{z}_l]\, x^l$$

with $z_i \in A[\xi \mid y]$, where $\bar{z}_i$ is the residue class of z_i in $\bar{A}[\xi \mid y]$. Note that $\delta' \bar{z}_i = 0$. Hence there exist $w_0, w_1, \cdots, w_l$ in $A[\xi \mid y]$ such that $\delta' z_i = \rho_1 w_i$, and so $\phi([\bar{z}_i]) = [\bar{w}_i]$. We see that $\delta' w_i = 0$ and put

$$v = z_0 + z_1 x + \cdots + z_l x^l$$

Then we have

$$\delta v = (\delta' + \rho_1 x)\, v = \rho_1 \{w_0 + (w_1 + z_0)\, x + \cdots + (w_l + z_{l-1})\, x^l + z_l x^{l+1}\}.$$

It follows that

$$\begin{aligned} \Delta([\bar{v}]) &= [\bar{w}_0] + [\bar{w}_1]\, x + \cdots + [\bar{w}_l]\, x^l + [\bar{z}_0]\, x + [\bar{z}_1]\, x^2 + \cdots + [\bar{z}_l]\, x^{l+1} \\ &= \phi([\bar{z}_0]) + \phi([\bar{z}_1])\, x + \cdots + \phi([\bar{z}_1])\, x^l + x\, [\bar{v}] \\ &= (\phi \otimes 1 + x\, id)([\bar{v}]), \end{aligned}$$

which proves our proposition.

Proposition 2. *There exists an polynomial $f(\xi \mid x, y)$ in $k[\xi \mid x, y]$ such that $f(\xi \mid x, y)$ is a monic polynomial in the variable x and $f\mathfrak{C}_A = 0$.*

Proof. Because $\mathfrak{C}'_{\bar{A}}$ is a finite $R[\xi \mid y]$-modules, we can choose a set $\{[\bar{u}_1], \ldots, [\bar{u}_t]\}$ of generator of $\mathfrak{C}_{\bar{A}}'$ with $\bar{u}_i \in \bar{A}[\xi \mid y]$. Then we can write

$$\phi([\bar{u}_i]) = \sum_{j=1}^{t} g_{ij}[\bar{u}_j] \qquad \text{with} \quad g_{ij} \in R[\xi \mid y]$$

It follows from Proposition 1 that

$$\Delta([\bar{u}_i]) = \phi([\bar{u}_i]) + [\bar{u}_i]x.$$

Let a matrix G be (g_{ij}). Then we have

$$\begin{pmatrix} \Delta([\bar{u}_1]) \\ \vdots \\ \Delta([\bar{u}_t]) \end{pmatrix} = (G - xE) \begin{pmatrix} [\bar{u}_1] \\ \vdots \\ [\bar{u}_t] \end{pmatrix}.$$

Let $adj(G - xE)$ be the adjoint matrix of $(G - xE)$. Multipling $adj(G - xE)$ we obtain

$$\mid G - xE \mid \begin{pmatrix} [\bar{u}_1] \\ \vdots \\ [\bar{u}_t] \end{pmatrix} = adj(G - xE) \begin{pmatrix} \Delta([\bar{u}_1]) \\ \vdots \\ \Delta([\bar{u}_t]) \end{pmatrix}.$$

Let $g(\xi \mid x, y) = \mid G - xE \mid$. we see that the leading term of $g(0,\ldots,0 \mid x, 0,\ldots,0)$ is x^t and that $g\mathfrak{C}_{\bar{A}}' \subset \Delta\ (\mathfrak{C}_{\bar{A}}'[x])$, and so $g\mathfrak{C}_{\bar{A}}'[x] \subset \Delta\ (\mathfrak{C}_{\bar{A}}'[x])$. Using the exact sequence in Theorem 1 we conclude that $g\mathfrak{C}_A = 0$, and let f be the residue class of g in $k[\xi \mid x, y]$ which satisfies conditions of our proposition.

Example. In Proposition 2 we cannot take x^t as f. Let k be a field and let η_1, η_2, η_3 be odd variables over k. Put

$$R = k[\eta_1, \eta_2, \eta_3]/(\eta_1\eta_2 - \eta_2\eta_3 + \eta_1\eta_3) = k[\rho_1, \rho_2, \rho_3],$$

where ρ_i is the residue class of η_i. The R is a local $\mathbb{Z}_2$-graded ring. Because $\rho_1\rho_2\rho_3 = \rho_1(\rho_1\rho_2 + \rho_1\rho_3) = 0$, we see that

$$R = (k + k\rho_1\rho_2 + k\rho_1\rho_3) + (k\rho_1 + k\rho_2 + k\rho_3)$$

$$m = k\rho_1\rho_2 + k\rho_1\rho_3,$$

and hence

$$m/m^2 = (m/m^2)_1 = k\rho_1 + k\rho_2 + k\rho_3.$$

It is easily seen that ρ_1 is an R-regular element. Let $\delta = \rho_1 x + \rho_2 y_2 + \rho_3 y_3$ and let $z = \rho_2\rho_3$. Then $\delta z = 0$, and hence $[z]$ is an element of $\mathfrak{C}_R$.

We will show that $x^n[z] \neq 0$ for all positive integer n. Indeed, assume that $x^n[z] = 0$ for some n. Then there exists an element g of $R[x, y_2, y_3]$ such that $zx^n = \delta g$. We write $g = wx^{n-1} + g_1$ with $w \in R$, where g_1 is a polynomial which does not contain the monomial term of the form ax^{n-1} with $a \in R$. Then we have

$$\rho_2\rho_3 x^n = (\rho_1 x + \rho_2 y_2 + \rho_3 y_3)\,(wx^{n-1} + g_1).$$

Comparing the coeffients of x^n, $x^{n-1}y_2$, $x^{n-1}y_3$, it follows that $\rho_2\rho_3 = \rho_1 w$, $\rho_2 w = 0$, $\rho_3 w = 0$. Put

$$w = a_1 + a_2\rho_1\rho_2 + a_3\rho_1\rho_3 + a_4\rho_1 + a_5\rho_2 + a_6\rho_3 \qquad \text{with} \quad a_i \in k.$$

Since $\rho_2 w = 0$ and $\rho_3 w = 0$, we know that $a_1 = 0, a_4 = 0, a_5 = 0, a_6 = 0$ and therefore $w = a_2\rho_1\rho_2 + a_3\rho_1\rho_3$. Now we get a contradition that $\rho_2\rho_3 = \rho_1 w = 0$. Finally we conclude that $x^n \mathfrak{C}_R \neq 0$ for all positive integer n.

We denote by $\mathfrak{a}(A)$ the ideal $(\mathrm{Ann}_{k[\xi \mid x,\,y]}\mathfrak{C}_A + (\xi))/(\xi)$ of the ring $k[x, y]$ and also denote by $\mathfrak{b}(\bar{A})$ the ideal $(\mathrm{Ann}_{k[\xi \mid y]}\mathfrak{C}_{\bar{A}} + (\xi))/(\xi)$ of the ring $k[y]$. Let Sing A be the close subset $V(\mathfrak{a}(A))$ in Proj $(k[x, y])$ and we call it the *singluar schme* of A. We define *odd depth* odepth A by codim Sing A. (Cf. 5.2 of [2])

We shall show an another proof of the following Theorem.

Theorem 2. [Theorem 5.4 in [2]] *Let A be a finite, non-zero module over a noetherian local ring R. Let ρ_1 be an odd and A-regular element. Then we have that* odepth A = odepth $A/\rho_1 A + 1$.

Proof. This theorem follows from the next three lemmata.

Lemma 4. odepth $A = \mathrm{ht}\,\mathfrak{a}(A)$ *and* odepth $\bar{A} = \mathrm{ht}\,\mathfrak{b}(\bar{A})$.

Proof. It is clear that odepth $A = \mathrm{ht}\,\mathfrak{a}(A)$. By Lemma 1 we have $\mathrm{Ann}_{k[\xi \mid x,\,y]}\mathfrak{C}_{\bar{A}} = (\mathrm{Ann}_{k[\xi \mid x,\,y]}\mathfrak{C}_{\bar{A}})[x]$ and thus $\mathfrak{b}(\bar{A})[x] = (\mathrm{Ann}_{k[\xi \mid x,\,y]}\mathfrak{C}_{\bar{A}} + (\xi))/(\xi)$. This implies that odepth $\bar{A} = \mathrm{ht}\,\mathfrak{b}(\bar{A})[x] = \mathrm{ht}\,\mathfrak{b}(\bar{A})$.

Lemma 5. $\mathfrak{a}(A) \cap k[y] = \mathfrak{b}(\bar{A})$ *and* $\mathfrak{a}(A) \neq \mathfrak{b}(\bar{A})[x]$.

Proof. To prove that $\mathfrak{a}(A) \cap k[y] = \mathfrak{b}(\bar{A})$ it is sufficent to show that for $f \in R[\xi \mid y]$, $f\mathfrak{C}_{\bar{A}} = 0$ if and only if $f\mathfrak{C}_A = 0$. It follows from Lemma 1 and Theorem 1 that $\mathfrak{C}_{\bar{A}}[x] \xrightarrow{\rho_1} \mathfrak{C}_A \longrightarrow 0$ is exact. This implies that if $f\mathfrak{C}_{\bar{A}} = 0$, then $f\mathfrak{C}_A = 0$. Conversely suppose that $f\mathfrak{C}_A = 0$. Let $[\bar{z}]$ be an element of $\mathfrak{C}_{\bar{A}}$. Becuase $\mathfrak{C}_{\bar{A}} \cong \mathfrak{C}'_{\bar{A}}$, we may assume that $[\bar{z}] \in \mathfrak{C}'_{\bar{A}}$. By Theorem 1, there exists $\bar{v}$ in $\bar{A}[\xi \mid x, y]$ such that $\delta\bar{v} = 0$ and $\Delta([\bar{v}]) = f[\bar{v}]$. As the proof of Proposition 1 we write

$$[\bar{v}] = [\bar{z}_0] + [\bar{z}_1]x + \cdots + [\bar{z}_l]x^l$$

with $z_i \in A[\xi \mid y]$. It follows from Proposition 1 that

$$\sum_{0 \le i \le l} f[\bar{z}_i]x^i = \sum_{0 \le i \le l} [\bar{z}_i]x^i + \sum_{0 \le i \le l} [\bar{z}_i]x^{i+1}.$$

Comparing the degree in x we see that $[\bar{v}] = 0$ and so $f[\bar{z}] = 0$, and therefore $f\bar{\mathfrak{C}}_{\bar{A}} = 0$.

By Proposition 2 there exists an element f in $\mathrm{Ann}_{k[\xi \mid x, y]}\mathfrak{C}_A$ such that f is monic in the vairable x. Then the residue class $\bar{f}$ in $k[x, y]$ of f belongs to $\mathfrak{a}(A)$. But $\bar{f}$ does not belong to $\mathfrak{b}(\bar{A})[x]$. Indeed, on the contarary assume that $\bar{f} \in \mathfrak{b}(\bar{A})[x]$. Then we can easily varify that $1 \in \mathrm{Ann}_{k[\xi \mid y]}\bar{\mathfrak{C}}_{\bar{A}}$, whence $\bar{\mathfrak{C}}_{\bar{A}} = 0$. By the next lemma $\bar{A} = 0$, hence $A = 0$ for ρ_1 is A-regular. Accordingly we get a contradiction and that proves our lemma.

Without assumption that ρ_1 is an A-regular element, we have the following lemma.

Lemma 6. *Let R be a noetherian local $\mathbb{Z}_2$-graded ring and let A be a graded R-module. If $\mathfrak{C}_A$ = 0, then $A = 0$.*

Proof. Let $R = R_0 + R_1$. Because R_1 is a finite R_0-module, A is a finite R_0-module and hence $\bigcap \mathfrak{m}_0^t A = 0$. We claim that if $\mathfrak{C}_A = 0$, then $0 :_A R_1 = 0$. Indeed, assume that $a \in A$ with $R_1 a = 0$. Since $\rho_i a = 0$ for all i, $\delta\,(a\xi_1\xi_2 \cdots \xi_p) = 0$. By assumption there exists an element $g \in A[\xi \mid x, y]$ such that $a\xi_1\xi_2 \cdots \xi_p = \delta g$. Put

$$g = \sum b_i \xi_1 \cdots \hat{\xi}_i \cdots \xi_p + h,$$

where h does not contain any monomial of the form of $c\xi_1 \cdots \hat{\xi}_i \cdots \xi_p$. It follows that

$$a = \sum (-1)^{i-1} r_i b_i' \quad \text{and} \quad \rho_i b_j' = 0 \qquad \textit{for all} \quad i\ j$$

with $b_i' = b_{i0} - b_{i1}$, where we write $b_i = b_{i0} + b_{i1}$ with $b_{i0} \in A_0$ and $b_{i1} \in A_1$. Whence $a \in \mathfrak{m}_0 A$ and $b_j' \in 0 :_A R_1$ for all j. Repeat the same procedure. Then we see that $b_j \in \mathfrak{m}_0 A$, and so $a \in \mathfrak{m}_0^2 A$. After iterating we get $a \in \bigcap \mathfrak{m}_0^t A$, so $a = 0$.

Let $a \in A$. Then $\rho_1\rho_2 \ldots \rho_q a \in 0 :_A R_1$. We can easily see that by induction on t that $\rho_{i_1}\rho_{i_2} \ldots \rho_{i_t} a = 0$ for all t, $1 \le t \le q$. In paticular $\rho_i a = 0$ for all i, therefore $a \in 0 :_A R_1$. We obtain that $a = 0$ and this proves that $A = 0$.

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