Structural Equations Formulated in the Endogenous Growth Model*

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This paper explains structural equations in the endogenous growth model. These equations are composed of four parts: (1) basic equations that exist in national accounts, (2) equations that connect the data of national accounts with those of the Cobb-Douglas production function, (3) three endogenous parameters hidden in the Cobb-Douglas production function, and (4) variables in reduced forms that are measured endogenously, some of which are shown in the process of (3). The author advocates that 'endogenous' will hold without partial approaches or econometrics and, consistently in the short and long run, where data and model go together. The author earlier started with an exogenous model of Robert Solow (1956). The author now confirms that a set of endogenous equations in this paper conscientiously follows what Solow (2008) implicitly intends to express. The author, however, stresses that Solow's one sector model be first divided into the government and private sectors. In the near future, the

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private sector may be endogenously divided into several sub-sectors so as to match each character of industries.

1. Philosophy of the endogenous growth model and definitions of terminologies

Before starting, the author will briefly state a philosophy behind and a version of the author's model as the endogenous growth model, and also definitions of terminologies used in this paper.

First, the philosophy and version to the endogenous growth model are summarized as follows: the author's philosophy is global policy-making by country and by fiscal year towards peaceful economic growth, with harmonious earth preservation and avoiding enlarging inequality. The author ideally respects a common basic idea of 'the optimum' in the literature that is specified to consumption optimum and consumers' behavior. Nevertheless, the author stresses, for its methodology, that consumption and economic growth should be integrated as a system towards sustainable growth in the long run, say at least fifty years ahead. In this respect, the author defines 'the optimum condition by country' such that in the long run to attain sustainable growth by *the change in the initial actual parameters by fiscal year*. The background is this: if the initial two parameters at the beginning of a fiscal year (the ratio of net investment to output and the growth rate of population) are given, the initial three theoretical parameters (the relative share of capital, the capital-output ratio, and the capital-labor ratio) are measured by fiscal year, setting basic data of national accounts consistently with the data used for the Cob-Douglas production function by fiscal year in the long run. Then, using these five parameters, three parameters implicitly included in the Cobb-Douglas production function (*beta*, *delta*, and *lambda*) are measured by fiscal year and also by recursive year *in the transitional path of a fiscal year*. The transitional path by fiscal year shows how all the parameters and variables of the model change by recursive year, by using the structural equations in the discrete time.

The initial data are commonly used in a fiscal year and in the first year of recursive years, where initial and current mean the same expression but *initial* emphasizes *initial* determinants to policy makers. In recursive years of the transitional path, the data (both parameters and variables) at the first year and the data at convergence year are calculated by using structural equations of the model (without relying on recursive programming). The optimum condition is determined by the combination of related parameters so that the initial two given parameters present a key to obtain the optimum condition with the three measured parameters. The endogenous growth model will realize the optimum condition by intentionally changing the initial parameters using policies pertinent in the long run (ironically but, maintaining a low optimum net investment). And, the convergence (situation) in recursive years is defined as a steady state that related variables have the same growth rates as in the literature, apart from exogenous or endogenous.

Strictly speaking, due to endogenous characteristics of the model, 'convergence' differs from the steady state. The author defines 'endogenous' such that a set of structural equations remain unchanged in the long run just like physics. Convergence in this paper only indicates the convergence in recursive years, which differs from the convergence in the literature. The convergence in the literature is used in fiscal years, e.g., by comparing the average growth rate of per capita GDP with the first year's per capita GDP using panel data among countries or panel data of a country. A definite reason is that there in the literature have not found endogenous/theoretical equations as in physics. Econometrics is based on try and error plausible equations that still differ from generality of physics. When convergence speed or the years for convergence is measured in the literature, its implication is much less strict than the convergence in this paper.

Let the author explain the above summary using definitions in detail. The optimum condition is not limited to consumption alone and sinks wholly into the structural equations. The optimum condition is determined according to each level and its combination of wholly related parameters so that sustainable growth is guaranteed with steady consumption and mitigating inequality. First, for *the changes in the initial actual parameters by fiscal year*, a country will look for and approach the optimum condition for people, by executing urgent policies by fiscal year. An aspect of the optimum condition, however, is rather compulsively given in the economic stage, which promotes the transition of poor, developing, and developed stages. This is the capital-output ratio at the developed stage. At the developing stage, the capital-output is free from compulsiveness, since the capital-output ratio is lower than that at the developed stage. When a country reached the developed stage due to a high level of net investment by fiscal year, its (theoretical) capital-output ratio cannot rise beyond an upper limit, say 2.5, under global competition.

This limit is proved by using the relationship between technological progress and the capital-output ratio in the structural or reduced form of related equations. If the capital-output ratio of the total economy rises highly beyond this limit, partly due to extreme deficits and debts in the government sector, the rate of technological progress is difficult to maintain a moderate level in the long run. Even if the private sector has a high rate of technological progress, the rate of technological progress of the government sector will offset the efforts of the private sector. This implies that fiscal policy implicitly obeys global competition by country. Therefore, if a country could get into one of developed countries, this country first of all must maintain the capital-output ratio as low as possible. This implies that the upper limit of 2.5 is a yardstick to optimum on average among countries. This is a reason why some countries still enjoy a moderate rate of technological progress in the long run even after reaching the developed stage. Furthermore, when the capital-output ratio is low, consumption increases most steadily allowing for a comparatively low level of the relative share of capital. In this respect, the EMU of 3% deficit and 60% debt each to GDP is a remarkable criterion that absorbs the spirit of the optimum condition as well (see Kamiryo, IAEC, Warsaw, 2008c, that proved the background using the endogenous growth model).

Second, *in the transitional path to recursive years at a fiscal year*, once given the initial five parameters, there is no room for manipulating an optimum condition by country, since each country's economic, fiscal, and financial policies were already determined in the last fiscal year and these are reflected in the above initial five parameters. National taste differs by the level of the propensity to consume that changes only by fiscal year (never by recursive year), where national taste expresses the relationship between consumption and saving at the macro level by country. Starting with national taste, the structural equations (and/or reduced forms of equations) are wholly related to the optimum condition between sustainable growth and mitigating inequality in the long run.

Two initial given parameters are the ratio of net investment to output, $i \equiv I/Y$, and the growth rate of population/employed persons, $n \equiv (L_1 - L_0)/L_0$ in the discrete time, where 1 and 0 show each fiscal year. Three measured parameters are the relative share of capital, $\alpha \equiv \Pi / Y$, the capital-output ratio, $\Omega \equiv K/Y$, and the capital-labor ratio, $k \equiv K/L$, where $\alpha \equiv \Omega \cdot r$. The values of α and Ω are theoretically measured at the change in the initial actual parameters by fiscal year, after clearing the matching and smoothening tests by fiscal year, where data and model match in the long run. The relative share α varies in the changes of the initial actual parameters by fiscal year while the α in the transitional path at a fiscal year remains unchanged by recursive year. Since these five parameters determine all the variables, the key to maintain stable economies is to change these five parameters at the beginning of a year, by using useful economic, fiscal, and financial policies and converting physical capital to human capital based on supreme spirit of human and earth life towards worldwide peace and environments.

Then, how is the optimum condition clarified in the above five parameters? The optimum condition will be structurally found from the relationship between each of the five initial parameters and the three endogenous parameters by fiscal year. The author will clarify this relationship in detail separately from this paper, to reveal the optimum condition more generally. Conclusively speaking, the ratio of net investment to output must be roughly 10–12% (not above 15%) in the long run and the growth rate of population must be a little bit plus, within an upper limit of the capital-output ratio, by restricting the range of deficits and debts so that the financial assets do not stir up the real assets existing as a firm base of the world economy. For example, if the ratio of money to output ($m = M/Y$) stays close to the level of the capital-output ratio, bubbles do not occur every decade, where the worldwide cooperation of the central banks must control money stock, similarly to the EMU rule to deficits and debts. This implies that the financial assets backed up by market principles should be a supplementary means to strengthen sustainable growth in the real assets. The deflation/inflation rate and assets-deflation/inflation rate in the real assets constitutes an aspect of the author's model and will be discussed separately from this paper, using the ratio of the rate of return to the wage rate and the valuation ratio.

Next for each definition in detail, the author first of all defines the endogenous growth model as a model that uses structural and reduced equations and remains unchanged over centuries just like the case of physics. The author, thus, uses 'endogenous' in the most narrow sense. All the models in the literature are not 'endogenous' in this sense. The endogenous growth model, first of all, measures the rate of technological progress by fiscal year and also by recursive year in the transitional path at a fiscal year. *The transitional path to recursive years at a fiscal year* (hereafter the transitional path) is defined as the path that shows how parameters and variables of the model change by recursive year, starting with the initial situation at recursive year $t=1$, where two parameters (n, α) remain unchanged and, three parameters (i, Ω, and k) change while measuring a set of three parameters (see the next paragraph) hidden in the Cobb-Douglas production function by recursive year. Note that the initial 'I' is fixed but 'i_t' by recursive year changes since output changes by recursive year. If the rate of technological progress is externally given in the transitional path by recursive year, the model is 'exogenous' in the author's definition.

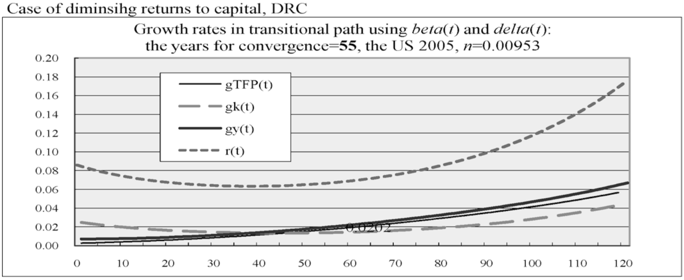
For example, if the rate of technological progress is measured using such as 'learning by doing' (Kenneth J., Arrow, 1962) and 'R & D' (Paul M., Romer, 1990), the model is exogenous. 'Education' (Robert E., Lucas, 1988) is most representative among representative endogenous models. No one denies that when education is vividly alive in labor and capital the rate of technology is sustainable in the long run. 'Learning by doing,' 'R & D,' and 'education,' to the author's understanding, cannot explicitly be traced back to technological progress: these aspects are not specified in the three hidden parameters (*beta*, *delta*, and *lambda*) that change by recursive year in the model. These aspects are only indirectly absorbed into the related parameters so that the above example models are not endogenous in the strictest sense. This is because all the flows and stocks are each a mixture of quality and quantity, which is divided into quality and quantity only by using *beta*.[1] Besides, the endogenous model in this

1) This paper does not explain for simplicity but, for example, the propensity to consume is divided into qualitative and quantitative propensity by fiscal and recursive year, and labor or capital (each stock) is divided into qualitative and quantitative labor or capital by fiscal and recursive year, using beta, where the above learning by doing,' 'R & D,' and 'education' are wholly spread over.

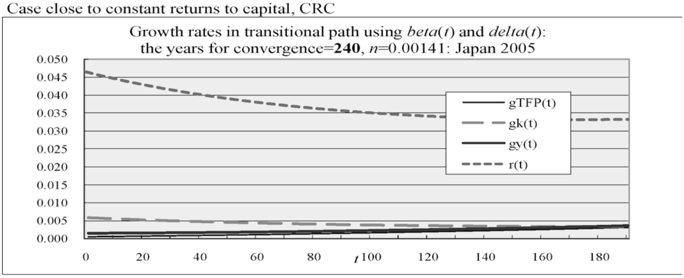
sense requires its necessary condition that the data of national accounts are consistent with the data used for the Cobb-Douglas production function by fiscal year in the long run, where the equivalent of three aspects exists: output = expenses = income. The necessary condition is guaranteed by satisfying the matching test and the smoothening test for national accounts data by fiscal year for at least thirty to forth years in the past (see the next section below).

For *the transitional path to recursive years at a fiscal year*, the author principally confines this model in the Cobb-Douglas production under constant returns to scale. Then, diminishing returns to capital (DRC) turns to constant returns to capital (CRC) at the point of time of convergence, where DRC and CRC are each measured by the rate of return (returns divided by capital) by recursive year. *Convergence* is defined as the situation that the growth rates of output and capital are the same and constant in the transitional path, where convergence may spread for some recursive years if DRC is mild. This definition in the endogenous growth model differs from the definition of the steady state in the textbooks. In the Cobb-Douglas production function, DRC converges to CRC according to the speed of convergence measured by country. In the author's model, however, increasing returns to capital (IRC) exceptionally holds starting at the initial situation and turns to CRC at the point of time of convergence. Why does IRC hold under constant returns to scale? IRC and a minus growth rate of population works for constant returns to scale by recursive year; IRC is justified only at the sacrifice of a given minus growth rate of population (see **Figure 1**). In the literature, IRC holds by removing constant returns to scale, which is moderate. The author (JES 11 (2), 64, 2008) also proved IRC by transforming the production function and using total factor productivity, but this paper focuses the Cobb-Douglas production function. This is because the marginal productivity of capital and the marginal productivity of labor are each connected with the rate of return and the wage (note returns and wages are theo-

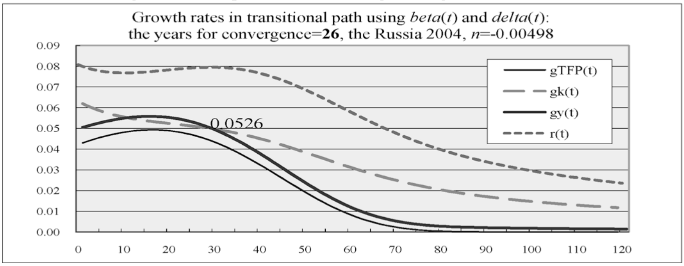
Case of diminsihg returns to capital, DRC



Case close to constant returns to capital, CRC



Case of increasing returns to capital (before reaching convergence)



Note: DRC, CRC, and IRC are defined by the rate of return in the transitional path, in particular before reaching convergence measured by the years for convergence. If DRC holds before at convergence, DRC turns to IRC after convergence. Similarly, If IRC holds before at convergence, IRC turns to DRC after convergence. If CRC holds before and after convergence, it is CRC throughout the transitional path. Even using the Cob-Douglas production function under constant returns to scale, IRC holds at the sacrifice of a minus growth rate of population.

Figure 1 Examples of DRC, CRC, and IRC before reaching the point of time of convergence

retically measured) under constant returns to scale.

The author distinguishes 'the transitional path to recursive years at a fiscal year' with 'the changes of the initial actual parameters by fiscal year,' where the economic stage hopefully shifts from poor to developing, and from developing to developed, in the long run. For each of the transitional path and the economic stage, the author's model commonly starts with the initial/actual/current situation. First, the initial/current situation has the transitional path to recursive years at a fiscal year, where the current situation gradually moves by recursive year and reaches the point of time of convergence. Convergence only holds in the transitional path. In this sense, this convergence differs from the convergence in the literature, which is observed in the economic stage in the short or long run in the past. The speed of convergence determines the point of time of convergence using three new parameters, *beta*, *delta*, and *lambda*. After the point of time of convergence, however, the situation moves further by recursive year. If DRC turns to CRC before convergence, CRC turns to IRC after convergence by recursive year. If IRC turns to CRC before convergence, CRC turns to DRC after convergence (see Figure 1). The literature does not clearly specify the convergence after convergence. This is because the convergence in the literature is explained by using the capital-labor ratio instead of recursive years.

Second, in the changes of the initial actual parameters by fiscal year, the current situation at the end of a fiscal year moves to the next current situation after one year: for example, consecutively 2005, 2006, 2007, and 2008. If the length of fiscal years is five to ten years, it is called 'in the short run,' and if the length is thirty to forty years, it is called 'in the long run.' Each year has its transitional path by recursive year, where the author does not use the terminology of 'in the short run' or 'in the long run.' Instead, the author uses the speed of convergence or the years for convergence in the transitional path. 'Forecasting' in the literature is used in the changes of the initial actual parameters by fiscal year, by extending the past trend to the future. When forecasting is taken into the transitional path, forecasting is replaced by the transition by recursive year, where ex-ante= ex-post, assuming the two initial given parameters are fixed throughout the transitional path. Ex-ante= ex-post implies that when econometrics methodology is used in parallel to the data by recursive year in the transitional path, its correlation coefficient will be 1.0. The structural equations in this paper and its data in the transitional path by recursive year hold without using econometrics yet the author, to find strict hypotheses, does not deny the use of econometrics approach to the data obtained in the changes of the initial actual parameters by fiscal year.

Now, what is the definite difference between the above convergence in the endogenous growth model and the steady state in the literature? Assume that the growth rate of population is plus and that DRC is close to CRC in the transitional path or the speed of convergence is enough long (say, the years for convergence are more than 100 years). Then, the steady state is well expressed in the endogenous growth model by country. This finding is shown in Figure 1. In the literature, convergence is only observed by using fiscal years in the past by country and/or among countries.

2. Basic equations that exist in national accounts

Basic data items in national accounts the author uses are consumption, C, saving, S, net investment, I, the balance of payments, $S-I$, and national disposable income (NDI), $Y \equiv C+S$. If the balance of payments is zero, it shows a closed economy while it is plus or minus, it shows an open economy. By taking into consideration the balance of payments, the equilibrium at the macro level holds by fiscal year, where supply equals demand by country. Related ratios are the propensity to consume, $c \equiv C/Y$, the propensity to save, $s \equiv S/Y$, where s = 1–

c or $c = 1-s$. This framework differs from that in the literature. For example, George Crowther (1957) and Paul De Grauwe (2005) show each representative framework using *GDP*, while the author's framework uses national disposable income as an endogenous base. Then, 'equivalent of three aspects,' first designed by Meade, J. E. and Stone, J. R. N. (1969, 344–345) holds, where Y = output = expenses = national disposable income.

The relationship between known/given and unknown data in the above framework is summarized as follows: Given actual data are the balance of payments and net saving as a base. Then, net investment is derived by $I = S-(S-I)$. If net saving is unknown, the propensity to consume, $c \equiv C/Y$, and consumption C must be given. Then, Y is derived by $Y = C/c$, where saving is derived: $S = Y-C$ by $Y \equiv C+S$. Consumption, saving, and national disposable income constitute a base of all accounts.

$$Y = C/c \text{ and } S = Y-C. \tag{1}$$

In statistics, additional items in detail and also statistical discrepancies stir up the simple form of $Y \equiv C+S$. Nevertheless, the model in this paper strictly sticks to $Y \equiv C+S$. As a result, data and model (including all the parameters and variables by sector) are consistent by fiscal year in the long run. The criterion of these tests is the maintenance of equilibrium by year and by sector and also the smoothness of business cycle with less depression. This criterion never contradicts optimum policies in a broad sense. This is the implication of the consistency of data and model under the endogenous growth model.

By the rule of aggregate/sum, an item value of the total economy is the sum of the corresponding value of the government sector and that of the private sector. For example, the balance of payments of the total economy, $S-I$, is the sum of budget surplus/deficit of the government sector, $S_G - I_G$, and the difference of private saving and private net investment, $S_{PRI} - I_{PRI}$, where $S = S_G + S_{PRI}$ and $I = I_G + I_{PRI}$ holds. The literature does not relate deficit $S_G - I_G < 0$ to government wages and returns due to the use of GDP. The author advocates, from the viewpoint of sub-system (see AAA, Anaheim, Aug, 2008) the importance of distinguishing the government sector with the total economy. The author also appeals the importance of comparing actually paid values such as wages/compensation and taxes with those theoretical in the author's model. These are explained briefly as follows:

In the government sector by fiscal and recursive year, wages equal consumption and returns equal saving simultaneously (see the labor function of consumption at the government sector below). Government output equals the actual sum of consumption and saving, where government output equals taxes as the theoretical sum of wages and returns. However, if government returns are minus, theoretical taxes decrease by the minus returns and minus saving. This will cause assets-deflation and decrease the government share of output. Then, what is the difference between the taxes in the model and the taxes actually paid in the SNA? People and policy makers must be alert at this difference to examine the performances of government activities. Likewise, in the total economy, for example, what is the difference between the theoretical wage rate in the model and the wage rate actually paid in the SNA? In recursive years, there is no wage inflation under a fixed relative share of labor. However, if the wage rate actually paid is more than the theoretical wage rate by fiscal year, there is a room for wage inflation. The theoretical wage rate exists completely in the real assets while the wage rate actually paid is influenced by financial assets to some extent due to financial policies. The author just indicates here the importance to compare the data of the SNA with the data used in the endogenous growth model.

The author's framework differs from the two sector model (divided into production/capital and consumption goods) as shown in Hirofumi Uzawa (1959, 1964). The author's framework basically formulates the equations using the total economy but these equations are applied to those of the government and private sectors each as well. The two (government and private) sector model in this paper will basically be an endogenous version of Solow's (1956) one sector model, where the author divides the total economy into the government and private sectors, each sector divided into consumption and saving.

3. Equations that connect national accounts with the Cobb-Douglas production function

Data and model must be consistent between wages, returns and capital (stock) by fiscal year. Wages, W, and returns, Π, are to be theoretically measured based on national disposable income by fiscal year: $W + \Pi = Y$. Otherwise, the relative share of capital, $\alpha \equiv \Pi / Y$ and the relative share of labor, $1-\alpha \equiv W / Y$, are still unknown under the equivalent of three aspects. Macroeconomics has its own characteristics that differ from those in microeconomics as advocated by Robert Solow (2008, p. 244), where the Ramsey's model does not directly hold. The author uses the consumption coefficient, (ρ/r), where ρ is the discount rate of consumption and r is the discount rate of wages each for people of the total economy that is composed of the two sectors at the macro level. The underlying idea is that the present value of consumption that uses ρ *equals* the present value of wages that uses r each in the infinite time (in recursive years). *The equality of the two present values* leads to Eq. 2 formulated below. Theoretical wages are originally used for consumption and saving, but in Eq. 2 consumption and theoretical wages are determined at the same time. The preference is whether people of the total economy consume or save, which is shown by ρ/r. When people want more consume than wages, $\rho > r$ may hold at the sacrifice of saving, where returns and saving turn to minus values. Thus, the theoretical relative share of labor $1-\alpha$ is formulated by (Kamiryo, 2005a):

$$1-\alpha = c/\left(\frac{\rho}{r}\right). \tag{2}$$

(ρ/r) is another expression of utility at the macro level and the author calls (ρ/r) 'national taste.' (ρ/r) shows a benchmark of the preference between consumption and saving. In particular, when the consumption coefficient, (ρ/r), is 1.0 as set in the government sector, consumption equals wages and saving equals returns, regardless of whether saving is plus or minus. (ρ/r) differs by fiscal year while it is constant in recursive years of the transitional path. Saving is composed of government saving, undistributed profits, saved dividends, and households saving: saving is the sum of government saving and private saving. Wages and returns are the sources of consumption and saving in the real world, but once consumption and saving are actually given, theoretical wages and returns are derived in the model, by using Eq. 2. The upper limit of the propensity to consume, $c = C/Y$, will be a little less 1.0 in the long run since net investment cannot be zero. However, from the viewpoint of sustainable growth and the optimum consumption, the propensity to consume will be specified roughly at 90% in the long run and under an open economy. Growth becomes sustainable when growth and inequality are balanced at the ratio of net investment to output i = 10% (as shown in Eq. 8 below).

It is possible to 'calibrate' the correlation coefficient R^2 (between unknown $1-\alpha$ and actual c) to be closer to 1.0, by changing the value of ρ/r at the Y axis to the value of the propensity to consume at the X axis by fiscal year and in the long run, consecutively 30 to 50 years. This is exceptionally a calibration in the author's data and model. Under global competition by country, a common labor/wages function of consumption is 'calibrated' among many countries, except for some countries that adopt specified tax systems to people as in Singapore and Malaysia. Then, the difference of national taste or preference under globalization mostly depends on the difference of the propensity to consume and the marginal propensity to consume by fiscal year that varies by country influenced by different economic, fiscal, and financial policies. Note that the government sector is neutral to the preference so that the author sets $\rho/r = 1$ to the government sector. This implies that government consumption equals government returns, where $\rho_G = r_G$. Then at the total economy, using theoretically derived $1-\alpha$,

$W = (1-\alpha)Y$ and $\Pi = \alpha \cdot Y$ hold. (3)

Referring to Robert Solow's (1958) "A Skeptical Note on the Constancy of Relative Shares," the author indicates that the relative shares are constant by recursive year in the transitional path while it changes by fiscal year in the economic stage in the long run.

Next, capital is measured using the capital-labor ratio, $k = K/L$, multiplied by labor or employed persons, L (Kamiryo, 2006a).

$$k = \frac{\alpha/(1-\alpha)}{\left(\frac{r}{w}\right)} \quad \text{or} \quad K = \frac{\alpha/(1-\alpha)}{\left(\frac{r}{w}\right)} \cdot L, \tag{4}$$

where r is the rate of return (that corresponds with the natural rate of interest) and w is the wage rate: $r = \Pi/K$ and $w = W/L$. Eq. 4 is an accounting identity, which differs from such a similar equation as shown in the two-sector model by Hirofumi Uzawa (ibid.). The two sector model and the literature usually use (w/r) as the reversed number of the above (r/w). Eq. 4 measures theoretical capital by year (using the matching test) and also tests the sustainability of the capital-output ratio in the long run by using the smoothening test. Under global competition, the capital-output ratio has its upper limit at the private sector, say 2.0–2.2. Because of this, capital should be consistently measured consecutively in 30–50 years, where no later adjustment is required. As a result, wages, returns, and capital are theoretically put into the Cobb-Douglas production function, where $\alpha \cdot W = (1-\alpha)\Pi$, $c \cdot S = (1-c)C$, and $K \cdot W\left(\frac{r}{w}\right) = L \cdot \Pi$ each hold. Note that stocks such as capital and labor are each composed of qualitative and quantitative elements which cannot be separable. In this respect, the author, at the earlier research towards 'endogenous,' avoided the use of physical and human capital, opposing Mankiw Gregory N., David Romer, and David N., Weil (1992).

4. Three hidden parameters found in the Cobb-Douglas production function

The Cobb-Douglas production function holds under constant returns to scale, assuming diminishing returns to capital (DRC) at the current situation and accepting Ken-ichi Inada's (1963) condition for K and L to become each zero or infinite.[2)] These presumptions are erased when the Cobb-Douglas production function reveals such implicit parameters, *beta*, *delta*, and *lambda*, as each measured below. Structural equations work completely by recursive year in the transitional path. Each parameter directly or indirectly needs initial/current five given and measured parameters: Two given parameters by recursive year are the ratio of net investment to output, $i \equiv I_0 / Y_0$, and the growth rate of population/employed persons, $n \equiv (L_1 - L_0) / L_0$, where L_0 is labor before a fiscal year and L_1 is labor after the fiscal year. Both i and n are constant by recursive year in the transitional path, but note that $i_t = i \cdot y_t$ changes by recursive year since y_t changes by recursive year. Three measured parameters by recursive year are the relative share of capital, $\alpha \equiv \Pi / Y$, the capital-output ratio, $\Omega \equiv K / Y$, and the capital-labor ratio $k \equiv K / L$, where $\alpha \equiv \Omega \cdot r$. In the transitional path, neverthe-

2) Barro, R. J., and Xavier Sala-i-Martin (1995, 16–17) summarizes the properties of the Cobb-Douglas production function and explains the Inada's condition with other properties. In the author's model, diminishing returns to capital (DRC) at the initial year holds in the transitional path and DRC is not an assumption. However, the literature will doubt increasing returns to capital (IRC) at the initial year in the transitional path, despite of constant returns to scale. This fact is justified by having a minus growth rate of population (see Eq. 9 below).

less, measured α is constant by recursive year, while measured Ω and r each change by recursive year. Along with the transition of the economic stage in the long run, the capital-output ratio has its upper limit under the smoothening test. From the optimum viewpoint, the minimum Ω implies the maximum r, assuming that the relative share of capital α is fixed as it is set so in recursive years. Therefore, if the capital-output ratio at convergence is set equal to the initial/current capital-output ratio, $\Omega_0 = \Omega^*$, the capital-output ratio at convergence is minimized. This presents an expression of the optimum condition in the broad sense.

The three hidden parameters, *beta*, *delta*, and *lambda*, are measured step by step in each reduced form. Let the author first summarize the relationship between these parameters. The first parameter, *beta*, is defined as the quantitative net investment to the sum of quantitative and qualitative net investment. *Beta* is expressed as $\beta(i, n, \alpha, \Omega)$. And, $1-$*beta* is the qualitative net investment to the sum of quantitative and qualitative net investment and expressed as $(1-\beta)(i, n, \alpha, \Omega)$. The author distinguishes the *beta* at the current situation (β_0) with the *beta* at convergence (β^*). β^* and $B^* \equiv \left(1-\beta^*\right)/\beta^*$ at convergence is measured immediately using an equation, while β_0 at the current situation is measured after measuring the second parameter of δ_0 at the current situation. And in the transitional path, *beta* by recursive year t, β_t, is calibrated in recursive programming, with the second parameter of δ_t and the third parameter of $1/\lambda$, based on $\beta_t(i, n, \alpha, \Omega_t, k_t, \delta_t, 1/\lambda)$, where Ω_t, k_t, δ_t, or $k_t^{\delta_t}$ are calculated by recursive year under a fixed speed of convergence, $1/\lambda$. The second parameter of δ_t neutralizes diminishing returns to capital (DRC) in the transitional path and realizes constant returns to capital (CRC) at convergence. The third parameter of $1/\lambda$ shows the speed of convergence or the years for convergence and *lambda* is the convergence coefficient (see Figure 1).

For beta in the transitional path:

'Endogenous' begins with the measure of *beta*. The value of β^* is measured, after formulating the growth rate of capital-labor ratio and the growth rate of per capita output in the transitional path. For explanation, the author starts with the increase in capital, ΔK, but ΔK is replaced by I, to distinguish the ΔK in the literature with the dividable net investment in the author's model.

β^* is defined as quantitative investment to total net investment: $\beta^* \equiv i_K / (i_K + i_A) \equiv i_K / i$, where $i \equiv I_0 / Y_0$, $i \equiv i_K + i_A$, $i_K \equiv i \cdot \beta^*$, and $i_A \equiv i(1-\beta^*)$, but $i \neq i_t$ (see below). β^* is derived as Eq. 8, using Eqs. 5, 6, and 7, starting with the capital-labor ratio, $k_t \equiv K_t / L_t$:

$$k_{t+1} \equiv \frac{K_t + \Delta K_t}{(1+n)L_t},\ k_{t+1} \equiv \frac{K_t + i_K \cdot Y_t}{(1+n)L_t} \text{ and accordingly, } k_{t+1} \equiv \frac{k_t + i_K \cdot y_t}{(1+n)}.$$

Then, $\Delta k_t = \dfrac{i_K \cdot y_t - n \cdot k_t}{1+n}$ using $\Delta k_t \equiv k_{t+1} - k_t$,

where $i_K \cdot y_t = i_t \cdot \beta^*$ due to $i_t = i \cdot y_t$ and $y_t \equiv Y_t / L_t$.

Therefore, the growth rate of per capita capital is (Kamiryo, 2005b):

$$g_{k(t)} = \frac{i_K \cdot y_t - n \cdot k_t}{(1+n)k_t} \text{ and } g_{k(t)} = \frac{1}{1+n}\left(i_K \cdot A_t \cdot k_t^{\alpha-1} - n\right) \text{ or } g_{k(t)} = \frac{1}{1+n}\left(\frac{i_K}{\Omega_t} - n\right),\text{[3)]} \tag{5}$$

where $\dfrac{1}{\Omega_t} = \dfrac{k_t^{1-\alpha} \cdot k_t^{\alpha-1}}{\Omega_t} = A_t \cdot k_t^{\alpha-1}$ since $A_t = \dfrac{k_t^{1-\alpha}}{\Omega_t}$ (see Kamiryo, 2003).

Using growth accounting, $g_{y(t)} = g_{A(t)} + \alpha \cdot g_{k(t)}$, in the Cobb-Douglas production function, as shown in the literature,

$g_y^* = g_k^* = g_A^* / (1-\alpha)$ holds at convergence. (6)

3) For the growth rate of capital, $g_{K(t)} \equiv \Delta K_t / K_t$: $\Delta K_t = I_{K(t)} = i_K \cdot A_t \cdot K_t^{\alpha} \cdot L_t^{1-\alpha} = i_K \cdot Y_t$.

Accordingly, $g_{K(t)} = i_K \cdot A_t \cdot k_t^{\alpha-1}$ and $g_{K(t)} = \dfrac{1}{1+n}(g_{K(t)} - n)$ hold.

Set Eq. 5 to be equal to Eq. 6, where Eq. 5 turns to the equation at convergence:

$$\frac{1}{1+n}\left(\frac{i_K}{\Omega^*}-n\right)=\frac{g_A^*}{1-\alpha}=g_k^*.$$

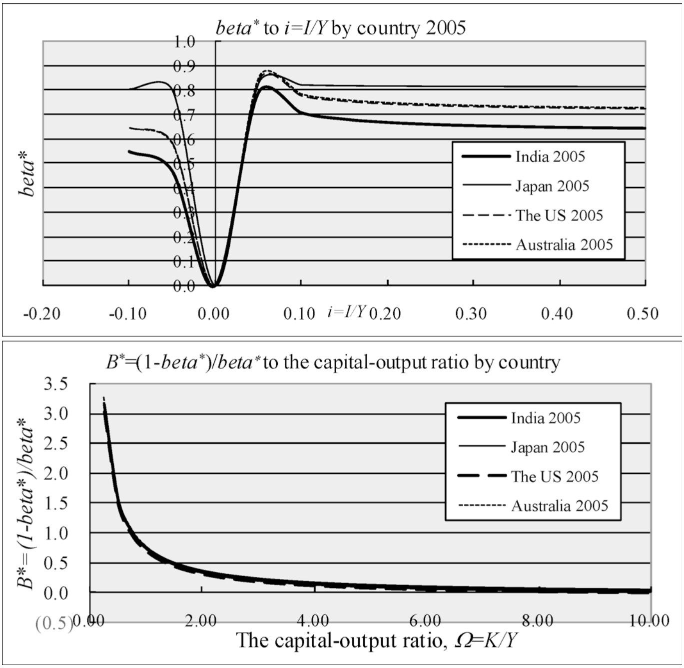
Thus, $\Omega^*=\dfrac{i_K}{\dfrac{i_A(1+n)}{1-\alpha}+n}$ or $\Omega^*=\dfrac{i\cdot\beta^*(1-\alpha)}{i(1-\beta^*)(1+n)+n(1-\alpha)}$. (7)

Therefore, by setting β^* on the LHS, $\beta^*=\dfrac{\Omega^*(n(1-\alpha)+i(1+n))}{i(1-\alpha)+\Omega^*\cdot i(1+n)}$, (8)

Note that optimum/minimum Ω^*_{MIN} at convergence is equal to Ω_0.

Eqs. 7 and 8 at convergence present good suggestions to approach the optimum condition and accordingly to people and policy makers. In particular, the level of the ratio of net investment to output at the initial situation, $i \equiv I_0/Y_0$, is closely related to the propensity to consume, with the balance of payments. The most useful relationships are derived using these equations. The examples by country using the author's data sets (KEWT 10.7, see Kamiryo, 2007b, c) are shown in **Figure 2.**

Next, the *beta* at the current situation and the *beta* in the transitional path are formulated likewise using the above process, except for the rate of technological progress, $g_{A(t)}$. The $g_{A(t)}$ in the transitional path is measured by introducing the relationship between *delta* (δ) and the relative share of capital (α). The δ is a parameter hidden in the Cobb-Douglas production function and changes diminishing returns to capital (DRC) to constant returns to capital (CRC) at convergence in the transitional path (for the equation, see the next section below). In the transitional path, if $\delta_t>\alpha$, the situation is under DRC and if $\delta_t<\alpha$, the situation is under increasing returns to capital (IRC). If $\delta_t=\alpha$, the situation is under constant returns to capital (CRC). The speed of convergence formulated by Robert Barro and Xavier Sala-i-Martin (1995) or earlier Xavier Sala-i-Martin (1990a, b) assumes DRC condition in the transitional path under an exogenous growth.



Note: Top of this figure shows that if the ratio of net investment to output is less than 10%, the quantitative net investment to total net investment $beta^*$ is unstable and that if it is more than 10%, the level of i does not influence $beta^*$ so much.
Bottom of this figure shows that if the capital-output ratio is more than 3.0, the qualitative to quantitative net investment at convergence B^* becomes extremely low, which implies that it is difficult to maintain the rate of technological progress at a moderate level. This characteristic indicates no difference between countries.

Figure 2 The relationship between the ratio of net investment to output and $beta^*$ and between the capital-output ratio and $B^* \equiv (1-\beta^*)/\beta^*$.

The rate of technological progress is defined as $g_{A(t)} \equiv (A_{t+1} - A_t)/A_t$ in the transitional path by recursive year. Using $\Delta A_t = (i_A \cdot y_t)/k_t^{\delta_t}$ or $\Delta A_t = A_t \cdot i_A \cdot k_t^{\alpha-\delta_t}$,

$$g_{A(t)} = i_A \cdot k_t^{\alpha-\delta_t}. \tag{9}$$

Similarly, using growth accounting, $g_{k(t)} = \dfrac{i_A}{(1-\alpha)k_t^{\delta_t-\alpha}}$, where under convergence $\delta_t - \alpha = 0$, and setting Eq. 5 to be equal to the above, $g_{k(t)} = \dfrac{i_A}{(1-\alpha)k_t^{\delta_t-\alpha}}$,

$$\beta_t = \frac{\Omega_t(n(1-\alpha)k_t^{\delta_t-\alpha} + i(1+n))}{i(1-\alpha)k_t^{\delta_t-\alpha} + \Omega_t \cdot i(1+n)} \text{ in the transitional path or,}$$

$$\beta_0 = \frac{\Omega_0(n(1-\alpha)k_0^{\delta_0-\alpha} + i(1+n))}{i(1-\alpha)k_0^{\delta_0-\alpha} + \Omega_0 \cdot i(1+n)} \text{ at the current situation.} \tag{10}$$

Note that for *beta* in the transitional path, recursive programming is required except for the *beta* at the current situation (β_0) and the *beta* at convergence (β^*), which are measured by using each equation. Finally, let the author define the qualitative net investment to quantitative net investment as $B \equiv (1-\beta)/\beta$. In this case, the B at convergence is measured as,

$$B^* = \frac{i(1-\alpha) + \Omega^* \cdot i(1+n) - \Omega^*(n(1-\alpha) + i(1+n))}{\Omega^*(n(1-\alpha) + i(1+n))}. \tag{11}$$

Eq. 11 is used for measuring *delta* as shown soon below.

For delta that leads DRC to CRC in the transitional path:

The value of *delta* by recursive year makes the Cobb-Douglas production function to hold consistently in the transitional path and without assumptions. Instead of using a few assumptions in the (exogenous) Cobb-Douglas production function, the author uses one assumption in the (endogenous) Cob-Douglas production function. This assumption is needed for measuring *delta* in the transitional path. The idea for formulating *delta* comes with endogenous measures of total factor productivity (A_t or TFP_t) in the Cobb-Douglas production function. The author (2008b) finds that TFP_t is not a 'residual' but measured by using β_t or B_t with the capital-labor ratio, k_t. Total factor productivity is completely measured at a flow level in the Cobb-Douglas production function. Total factor productivity is also measured at three stock levels by using B_t and k_t. However, these stock levels are not directly consistent with the Cobb-Douglas production function.

A base for measuring *TFP* in the Cob-Douglas production function comes from $TFP_t = A_t = k_t^{1-\alpha} / \Omega_t$ as an accounting identity as proved in the Coob-Douglas production function. The above equations (Eqs. 5 to 11) prove that *TFP* is not a residual. This proof is done by introducing Eq. 7 into $TFP_t = A_t = k_t^{1-\alpha} / \Omega_t$, where TFP_t is formulated using i, n, α, Ω_t, β_t or B_t, k_t, and δ_t, with the speed of convergence (soon below). As explained soon below, $1 \neq B^{*(1-\delta_0)} \cdot \Omega^*$ exists in the Cobb-Douglas production function. Nevertheless, the assumption of $1 = B^{*(1-\delta_0)} \cdot \Omega^*$ is required for measuring *delta* to erase all other assumptions in the (exogenous) Cobb-Douglas production function. The value of *delta* absorbs discrepancies in the Cobb-Douglas production function.

Explaining in detail, the above assumption, $1 = B^{*(1-\delta_0)} \cdot \Omega^*$, is required for determining the initial *delta*, δ_0. The LHS of this equation shows that $1 = k_t^{\delta_t - \alpha}$ holds at convergence, where $\delta_t = \alpha$ due to $\delta_0 \underset{t}{\rightarrow} \alpha$ at constant returns to capital (CRC). In the Cobb-Douglas production function, both $1 \neq B_0^{(1-\delta_0)} \cdot \Omega_0$ and $1 \neq B^{*(1-\alpha)} \cdot \Omega^*$ exist. These contradict $1 = B^{*(1-\delta_0)} \cdot \Omega^*$, and this is needed to solve the relationship between DRC/IRC and CRC under constant returns to scale. Once, δ_0 is measured with the speed of convergence, the transitional path from DRC/IRC to CRC is guaranteed by recursive year in the transitional path. Using the above equation of $1 = B^{*(1-\delta_0)} \cdot \Omega^*$ (Kamiryo, 2006b),

$$\delta_0 = 1 - \frac{LN(1/\Omega^*)}{LN(B^*)} \text{ is derived.} \tag{12}$$

Without this equation, the Cobb-Douglas production function remains exogenous.

Let the author show a different TFP_t as a stock level, for comparison. This case reduces to a Ak model by defining (see JES 11 (Feb, 2), 2008),

$TFP_t \equiv B_{TFP(1-\delta, 1-\alpha: FINAL)t}^{1-\delta_t} \cdot k_t^{1-\alpha}$. As a result,

$B_{TFP(1-\delta,1-\alpha:FINAL)t} = \left(\frac{TFP_t}{k_t^{1-\alpha}}\right)^{\frac{1}{1-\delta_t}}$. Note that $TFP_t = k_t^{1-\alpha}/\,\Omega_t$.

Then, $y_t = B_{TFP(1-\delta,1-\alpha:FINAL)t} \cdot k_t$ and $\Omega_{TFP(FINAL)t} = 1/\,B_{TFP(1-\delta,1-\alpha:FINAL)t}$.

This production function exactly shows an Ak model and corresponds with the AK model in Hussein Khaled and Anthony P. Thirlwall (2000). The author stresses that even the AK model in the literature is expressed endogenously by using related parameters such as β_t and δ_t, being released from the karma that total factor productivity is a residual.

For 1/lambda that controls the speed of convergence in the transitional path:

The speed of convergence shows how many recursive years an economy needs to reach CRC. This speed differs by country and by the economic stage. If the recursive years for convergence is short (e.g., 20 to 40 years) as in China and India, these countries remain at the developing stage while if it is enough long (e.g., 100 to 150 years as in Japan) the country stays at the developed stage. If DRC at the initial situation is significant, the speed of convergence is fast and if not the speed of convergence is slow. If CRC prevails by recursive year in the transitional path, it implies that the CRC situation continues in the infinite recursive years.

The difference between the exogenous and endogenous Cobb-Douglas production functions is typically shown by the speed of convergence. Endogenous implies that the rate of technological progress is completely measured within the Cobb-Douglas production function while exogenous, not. There are empirical papers that measure the speed of convergence, but based on panel data among countries and econometrics approaches. Nevertheless, the relationship between the exogenous speed of convergence and the endogenous one is tightly related. The endogenous speed of convergence replaces two items used in the exogenous speed of convergence with the rate of technological progress (Eq. 6) and *delta* (Eq. 12) (see, Kamiryo, 2006b).

Exogenous case: the convergence coefficient, $\beta = (1-\alpha)(n+x)$. (13)

Endogenous case: the convergence coefficient, $\lambda = (1-\alpha)n + (1-\delta_0)g_A^*$. (14)

Let the author explain the differences between Eq. 13 and 14. δ_0 in Eq. 14 corresponds with one 'α' of two in Eq. 13 that would appear under CRC. g_A^* in Eq. 14 corresponds with an exogenous rate of technological progress, 'x,' in Eq. 13. When $\alpha = \delta_0$ and $x = g_A^*$, both equations completely overlap. Also, the years for convergence of Eq. 13 are $0.69/\beta$, where $e^{-x}=0.5$ if $x=0.69$, assuming the half level to the situation of CRC, while the years for convergence of Eq. 14 are precisely calculated as $1/\lambda$, without any assumption.

Barro and Sala-i-Martin (ibid.), under DRC, measures the exogenous speed of convergence by assuming that the horizontal asymptote of DRC curve will hold at an infinite time of the transitional path (See Inada (ibid.)). If Barro and Sala-i-Martin (ibid.) could find the parameter of *delta* that is measurable, the difference between the two equations only comes from whether or not the rate of technological progress is endogenous. However, without an endogenous rate of technological progress, the value of *delta* cannot be measured. By these reasons, the author is able to take the same methodology to measure the speed of convergence as Barro and Sala-i-Martin's (ibid.) methodology. Dr. Toshimi Fujimoto (for his proof, see pp. 158–161, Appendix, Sep, 2006) helped the author to prove and review the formulation of Eq. 14. The results of the endogenous speed of convergence measured by country in 1960–2005 are much shorter than those of the exogenous speed of convergence in the literature.

After calculating the speed of convergence by country and by fiscal year, related parameters and variables are calculated by recursive year using recursive programming. By using structural and reduced equations formulated in this paper, related parameters and variables are directly measured only for the current/initial situation and at convergence. Related parameters and variable by recursive year in the transitional path (except for the current/initial situation and at convergence) must be calculated by using recursive programming. When a set of theoretical equations (which holds just like physics or mathematics) is used in the macro level equilibrium by year, calibration reduces to calculation. The author has been stimulated by Robert Solow's (pp. 243–246, Winter 2008, *JEP*) in that he indicted the importance of calibration in the macro level and that his whole version to the macro level really corresponds with the author's.

When the two parameters, *beta* and *delta*, use the same speed of convergence, each constant discount rate is calculated. For example, the distance between β_0 and β^* divided by the years for convergence determines the discount rate, $r_{CONVERGENCE(\beta)}$. Using LN in the Excel for convenience, where *ln* = LN (Kamiryo, 2006b),

$$r_{CONVERGENCE\,(\beta)} = \left(LN(\beta^*) - LN(\beta_0)\right) / \left(\frac{1}{\lambda}\right). \tag{15}$$

For confirmation, $r_{CONVERGENCE\,(\beta)} = POWER\left(2.7182818, \left(\dfrac{LN(\beta^*) - LN(\beta_0)}{1/\lambda}\right)\right) - 1$.

$$r_{CONVERGENCE\,(\delta)} = \left(LN(\alpha) - LN(\delta_0)\right) / \left(\frac{1}{\lambda}\right). \tag{16}$$

For confirmation, $r_{CONVERGENCE\,(\delta)} = POWER\left(2.7182818, \left(\dfrac{LN(\alpha) - LN(\delta_0)}{1/\lambda}\right)\right) - 1$.

Similarly, $$r_{CONVERGENCE\,(r/w)} = \left(LN\left(\frac{r}{w}\right)^* - LN\left(\frac{r}{w}\right)_0\right) / \left(\frac{1}{\lambda}\right). \tag{17}$$

Then, by recursive year (t) in the transitional path,

$$\beta_t = \beta_0 (1 + r_{CONVERGENCE(\beta)})^t. \tag{18}$$

$$\delta_t = \delta_0 (1 + r_{CONVERGENCE(\delta)})^t. \tag{19}$$

$$\left(\frac{r}{w}\right)_t = \left(\frac{r}{w}\right)_0 (1 + r_{CONVERGENCE(r/w)})^t. \tag{20}$$

Or, $\left(\dfrac{r}{w}\right)_t = \left(\dfrac{r}{w}\right)_0 (1+ g_y^*)^t$, where $g_y^* = r_{CONVERGENCE\left(\frac{r}{w}\right)}$.

Eq. 17 is used for the theoretical rate of return to the theoretical wage rate by recursive year assuming that this ratio is linear when the rate of return to the wage rate at convergence, $\left(\dfrac{r}{w}\right)^*$, is measured. And, if the wage rate at convergence, w^*, is measured, the capital-labor ratio at convergence, k^*, is measured, and accordingly, TFP* and y^* as well. When it is difficult to measure w^*, $r_{CONVERGENCE\left(\frac{r}{w}\right)}$ is replaced by g_y^* since g_y^* changes most steadily in the transitional path. Note that $r_0 = r^*$, which corresponds with $\Omega_0 = \Omega^*$.

5. Variables in reduced forms that are measured endogenously

Among variables, the growth rates at the current situation and at convergence were formulated already when the author formulated *beta* and *delta* above. This section formulates the rate of return and related variables, including the relationship between the rate of return, r^*, and the growth rate of output at convergence, g_Y^* (Kamiryo, 2007a).

$$r^* = \alpha\left(\frac{i(1-\beta^*)(1+n)+n(1-\alpha)}{i\cdot\beta^*(1-\alpha)}\right), \text{ where } r^* = \alpha/\Omega^*. \tag{21}$$

Note that $K = \Pi\sum_{t=0}^{\infty}\left(\dfrac{1}{1+r^*}\right)$ or $K=\Pi/r^*$ holds.

$$g_Y^* = \frac{g_A^*(1+n)}{1-\alpha}+n, \text{ by using } g_y^* = \frac{g_A^*}{1-\alpha}, \text{ or } g_Y^* = \frac{i\cdot(1-\beta^*)(1+n)}{1-\alpha}+n. \tag{22}$$

Let the author show the relationship between Eqs. 21 and 22. This relationship will reveal the difference between the author's endogenous golden rule and the exogenous golden rule established by Edmund Phelps (1961, 1966).

$$g_Y^* = \frac{\left(i(1-B_{g_Y^*})-A_{g_Y^*}\right)(1+n)}{1-\alpha}+n \text{ using } g_A^* = i(1-B_{g_Y^*})-A_{g_Y^*} = i(1-\beta^*),$$

where $A_{g_Y^*} \equiv \dfrac{i(1-\beta^*)(1+n)}{1-\alpha}$ and $B_{g_Y^*} \equiv \dfrac{\Omega^*(1+n)}{(1-\alpha)+\Omega^*(1+n)}$ are used just to offset some parameters.

When Eq. 22 is reformulated using the above equations, $A_{g_Y^*}$ and $B_{g_Y^*}$ the relationship between Eqs. 21 and 22 simply reduces to,

$$r^* = \left(\frac{\alpha}{i\cdot\beta^*}\right)\cdot g_Y^* \text{ or } g_Y^* = \left(\frac{i\cdot\beta^*}{\alpha}\right)\cdot r^*. \tag{23}$$

The author calls $\alpha/(i\cdot\beta^*)$ he Petersburg coefficient. Eq. 23 presents useful fundamentals to economic, fiscal, and financial policies. This is because, the three parameters of α, i, and β^* are able to control the relationship between r^* and g_Y^*. Furthermore, the above relationship is expressed by the cost of capital, $r^* - g_Y^*$, since the yearly flow of returns grows by g_Y^* under a constant relative share of capital: $\dfrac{1}{r^* - g_Y^*} = \sum_{t=0}^{\infty}\left(\dfrac{1+g_\Pi^*}{1+r^*}\right)$.

$$r^* - g_Y^* = g_Y^*\left(\frac{\alpha}{i\cdot\beta^*} - 1\right). \tag{24}$$

The cost of capital above is theoretical and measured by sector and differs from the user cost of capital, which is developed by Dale Jorgenson (1963) and Jorgenson and et al (1967). The user cost of capital uses econometrics with the market data only to the corporate sector.

The valuation value of capital, V, and accordingly, the valuation ratio, $v \equiv V/K$, are now measured by using the rate of return and the cost of capital, assuming that returns by recursive year is constant.

$$V = \Pi/(r^* - g_Y^*) \text{ and } v = r^*/(r^* - g_Y^*), \tag{25-1}$$

The valuation value Y differs from the market valuation value or fair value in that Eq. 25-1 is theoretically measured, not depending on the market in the short run. The valuation ratio, $v = \dfrac{r^*}{r^* - g_Y^*}$, is reformulated as a hyperbolic curve of

$v = 1 + \dfrac{g_Y^*}{r^* - g_Y^*}$, where the horizontal asymptote is 1.0 and the vertical asymptote is zero setting the cost of capital at the *X* axis (see **Figure 3**). This implies that when the cost of capital remains minus, the valuation ratio is below 1.0, after the valuation ratio turns to a plus from a minus. Note that the valuation value has both minus and plus values, if the cost of capital is minus close to the origin. The valuation ratio is maximized at both deficit = 0 and debt = 0, and then the valuation value decreases along with the cost of capital becomes positively higher. And, the valuation ratio never reaches 1.0, since it implies an infinite cost of capital, plus or minus. The condition for solvency in De Grauwe's (ibid, 225) corresponds with the above vertical asymptote, where the nominal interest rate equals a given growth rate of GDP.

The valuation ratio is, more importantly, expressed technologically or qualitatively since the Petersburg coefficient is shown by $\alpha/(i \cdot \beta^*)$:

$$v = \frac{\alpha}{\alpha - i \cdot \beta^*} = \frac{r^*}{r^* - r^*\left(\dfrac{i \cdot \beta^*}{\alpha}\right)} \text{ or } v = \frac{-\alpha / i}{\beta^* - \alpha / i}. \tag{25-2}$$

This implies that technological changes are definitely involved in the valuation ratio at convergence in the transitional path, where the initial α and the initial i

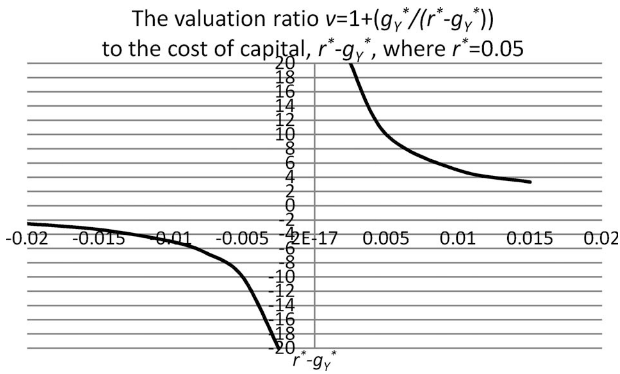


Figure 3 The hyperbolic curve of the valuation ratio to the cost of capital

are also used at convergence.

Technology is related to capital gains in the financial assets. Nicholas Kaldor (1961, 1966) and Tobin James and William C. Brainard (1977) discussed capital gains, which are now digested in the above Eq. 25-2. Capital gains are well justified by the valuation ratio and the theoretical capital, consistently measured in the long run (with the matching test and the smoothening test in the endogenous growth model). Eqs. 24 and 25-2 show the essence of the endogenous golden rule. As a result, the Petersburg paradox (historically discussed by David, Durand (1957)), where $r^* = g_Y^*$ and v = impossible, is avoided by taking urgent policies to change the related three parameters, α, i, and β^*, as shown in Eqs. 23 and 25-2.

However, in the case of the government sector, if national debt, D, is zero, v_G is maximized, if $D>0$ (which means national lending), $v_G>1.0$, and if $D<0$, the valuation ratio v_G usually lies between 0.55 and 0.95, depending on the level of budget deficit to national income, $\Delta d \equiv \Delta D / Y$. Whenever deficit exists, $V_G<K_G$ exactly holds in the government sector. And, this influences the valuation ratio of the total economy.

In this respect, is debt national equity/wealth (as raised by Robert Barro (1974))? Underlying equations at the stock level are shown as (Kamiryo, 2008c),

$$E = v \cdot K + D = V + D \text{ and } E_G = v_G \cdot K_G + D = V_G + D. \tag{26}$$

$$\text{For leverage } 1_{EV} \equiv -D/E,\ 1_{EV} = \frac{-D}{V+D} \text{ and } 1_{EV(G)} = \frac{-D}{V_G + D}. \tag{27}$$

If debts D are minus, it implies D reduces equity E and if debts are plus, it implies lending. When debts are more minus, the more $v_G<1.0$ and $V_G<K_G$ hold, resulting in a higher leverage due to its definition as above. Debt = 0 shows that v_G is maximized in the government sector. This proves that debts reduce equity.

In an open economy, v, v_G, and v_{PRI} changes by year. This is qualitatively traced back to the change in $1-\beta_G$, $1-\beta_{PRI}$, and accordingly, $1-\beta$, although qualitative elements occupy roughly 20–30% of the sum of qualitative and quantitative (see Eq. 9 and EQ. 25-2). Therefore, it is important to compare the β of the total economy with the β_G of the government sector at the current situation and at convergence. When the output share of the government sector is low (i.e., Y_G/Y is between 0.1 and 0.15 under a small government), the conditions of $\beta_G <$ β and $\beta_G^* < \beta^*$ often happen while the output share of the government sector is high (i.e., Y_G/Y is between 0.15 and 0.30 under a large government), the conditions of $\beta_G > \beta$, $\beta_G^* > \beta^*$, and/or $\beta_G > 1.0$ and $\beta_G^* > 1.0$ usually occur. Nevertheless, the conditions of $\beta_G < \beta$, $\beta_G^* < \beta^*$, and/or $\beta_G < 1.0$ and $\beta_G^* < 1.0$ sometimes occur, even if $v < 1.0$ is shown under a significant level of debts. These imply that the government sector is not monopolistic. For stable economic growth, $1-\beta_G$ must be compared with $1-\beta_{PRI}$, where $1-\beta$ of the total economy is the weighted average of two sectors. The literature that treats exogenous growth models has not discussed the comparison of these *betas* by sector.

Finally, the author connects the capital-output ratio, Ω, in the real assets with money supply (as a stock similarly to capital), M, in the financial assets. This is because the author sets real assets as a theoretical base of endogenous growth and because the author asserts that economic growth is stable and sustainable only when the trend of money matches that of the capital-output ratio in the long run.

$$\Omega_t = \Delta\Omega((Y_t - Y_{t-1})/Y_t) + \Omega_{t-1}(Y_{t-1})/Y_t) \text{ or,}$$

$$\Omega_t = \Delta\Omega \cdot g_{Y(BACKWARD)} + \Omega_{t-1}(1 - g_{Y(BACKWARD)}), \text{ where} \tag{28}$$

Notation of BACKBWARD is used for connecting the average with the marginal (For the first appearance of the relationship between the marginal and average, see Kamiryo, 1990).

$\Delta\Omega \equiv \Delta K / \Delta Y$, $g_{Y(BACKWARD)} \equiv \left(\dfrac{Y_t - Y_{t-1}}{Y_t}\right)$ and $g_{Y(BACKWARD)} \equiv g_Y / (1 + g_Y)$.

Paul Du Grauwe (ibid., 225) assumed marginal money stock is zero in formulating his condition for solvency of national debts: $dM/dt = 0$. To mitigate this assumption existing in the real assets, the author takes advantage of the level of money M and $m \equiv M / Y$, whose original idea comes from the Marshall's k (Milton Friedman, 1957).

$m_t = \Delta m \cdot g_{(BACKWARD)} + m_{t-1}(1 - g_{(BACKWARD)})$, where $\Delta m \equiv \Delta M / \Delta Y$. (29)

Furthermore, if the coefficient of neutrality in financial assets is defined as $c_{(\frac{M}{K})} = m / \Omega$,

$c_{(\frac{M}{K})t} = \Delta c_{(M/K)t} \cdot g_{K(BACKWARD)} + c_{(\frac{M}{K})t-1}(1 - g_{K(BACKWARD)})$, where (30)

$g_{K(BACKWARD)} \equiv \left(\dfrac{K_t - K_{t-1}}{K_t}\right)$ and accordingly, $g_{K(BACKWARD)} \equiv g_K / (1 + g_K)$.

The real assets work for equilibrium and the financial assets evaluate or supplement the real assets (as advocated by Robert E., Lucas, 1995), by the market principle and within the range that $\Omega \equiv K / Y$ is able to control $m \equiv M / Y$. In this sense, Eq. 29 is most important to policy-makers in the long run.

In terms of two types of agent-cutting directions towards a general equilibrium, Weidenbaum Murray (2008, pp. 248) pointed out that one starts with a single type of agent and boost it to three types while the other (i.e., Solow) starts with eight types of agents and cuts back to three types. For detail, see Solow (2004, p.661, 2008, p. 244) that cited Brainard William C., and James Tobin (1968). The author asserts that the above types of agent-cutting should be based on the real assets, starting with three agents (namely, the government, private, and total as an aggregate), where the differences between the real and financial assets are absorbed into the real assets or the balance of payments. Thereby agents in the financial assets are expressed using the real assets and national debts (see Eq. 26). This direction will be wholly justified by Robert Solow (pp. 243–246, 2008). The optimum condition of the real assets appears when Eq. 27 completely overlaps Eq. 28. If prices of products and services rise, the arbitrage in the market works well while if prices fall down the arbitrage does not work and, bubbles burst open every ten to twelve years as shown early in 2008. In this respect, some restrictions on money supply (after 1974) are directly required for stable growth comparing Eq. 27 with Eq. 28, similarly to the Maastricht Convergence Criteria for deficits, debts, inflation rates, and interest rates, as shown in the EMU rule, where 'endogenous' must be a base among countries.

6. Concluding remarks

This paper endogenously formulated a set of structural and reduced equations of the total economy in the real assets, with the proof to each equation. These equations are applicable to those in the government and private sectors as shown in the discussions on deficits and debts. The related equations guarantee the ex-post equilibrium by fiscal year in the short and long run, supported by the matching and smoothening tests and guaranteed by the three wage-rates growth test, where the economic stage changes by country. The three wage-rates growth test will be discussed in a coming paper that arranges the data-sets of KEWT 2.08, 1990–2006, to 58 countries by sector, similarly to KEWT 1.07, 1960–2005, to nine countries by sector (Kamiryo, 2007b, c). When a model is more general and long-oriented the more steadily it works, regardless of whether calibration, recursive programming, and/or econometrics are well designed or not. The model in this paper is the endogenous growth model in the narrowest sense in that each equation remains unchanged like physics in the natural world. The author understands that Solow (2008) suggested this direction.

Definite differences between exogenous and endogenous growth models in the literature and the author's model are: (1) The literature essentially needs econometrics while the author's model directly obtains theoretical equations and results such as the growth rates of technological progress, output/income, capital, per capita income and capital, and the rate of return and, by sector. (2) The literature assumes that government should not earn its returns, plus or minus, while the author's model separates government wages and government returns, cooperating with the total economy under the rule of aggregate. Note that for the test of hypotheses the author does not deny the use of panel data.

The endogenous growth model in the narrowest sense is flexible to cooperate with economic, fiscal, and financial policies by presenting several or more aspects. Some of these aspects are the relationship between the economic stage (Kamiryo, 2007d) and business cycle (Kamiryo, 2007e), the review of capital stock by sector, comparing it with capital stock in national accounts statistics (Kamiryo, 2007f), and the cooperation between the government sector with the private sector for earth environments and sustainability (Kamiryo, 2008a). These were already discussed in separate papers. For useful policies, it is also important to compare the actual data in national accounts with the theoretical data in the author's model. For example, in the literature the deflation/inflation rate has been discussed using the output-inflation tradeoffs (or reverse Phillips curve), indicating that if the variance of nominal output is more wide and unstable the relationship between the inflation rate and real output is less reliable (see Robert E., Lucas, 1973, 333–334). This issue will be solved when the magnitude of diminishing returns to capital (related to the growth rate of population, employed persons, and the unemployment rate) are analyzed using *delta* and *alpha*, where the speed of convergence differs significantly by country and by year. Also, the assets-deflation/inflation is only solved using the valuation ratio as in this paper, where deficits and debts must be restricted globally by country. And, the Marshall's k in the financial assets and the capital-output ratio in the real assets are closely related to the rate of technological progress (with elasticity analysis of the three parameters hidden in the Cobb-Douglas production function). These are all policy-oriented and will be discussed separately each by each.

The author's conclusion is that the endogenous and consistent data and model make it possible for an economy to find an optimum balance between sustainable growth and stop-inequality. This is because the endogenous rate of technological is deeply involved in sustainable consumption. When the economy is more human capital-oriented and wage-oriented (though it seems ironical), the capital-output ratio remains low and competitive globally (see Figure 2). By politically adjusting the relative shares of capital and labor (α and $1-\alpha$) using the wage/labor function of consumption (Eq. 2), consumption will be given a base for more qualitative. The parameter of $1–\beta$ prevails not only in net investment but also similarly in other items such as consumption and wages. Qualitative consumption and wages supported by philosophy of earth preservation are much more important than nominal growth of output, where growth shows a result only.

内生的成長モデルの性格（**Abstract**）

1. 外生的から（最狭義の）内生的成長モデルへ：マクロにおける外生的成長モデルと内生的成長モデルとの本質的相違は，技術進歩率が外から与えられるか（その嚆矢は，Solow, R. M., 1956），モデルのなかで測定できるかである。しかし，内生的なモデルであっても，広義と狭義とがある。文献上の内生的成長モデルは，learning by doing（Arrow, K. J., 1962), R & D（Lucas, R. E. Jr., 1988), そして，教育（Romer, P. M., 1990）のように，技術進歩率に強く結び付く項目を，モデル式のなかに入れて，内生的に技術進歩率を測定する。それらは，すべて広義である。最狭義の内生的成長モデル（以下，本モデル）は，会計年度ごとに，partial を排除して，general に成立する均衡モデルをデータと整合的に先ず整序し，そのモデルに含まれる内生パラメータを用いて，技術進歩率を完全に測定するようなモデルである。「データと整合的モデル」とは，国民経済計算制度（System of National Account of United Nations）に示されるソース・データがそのままコブ・ダグラス生産関数において必要とされるデータと，会計年度毎に，数十年にわたって年々整合的に維持できるというモデルである。それは生

産関数を用いないケインズ学派と生産関数を展開する新古典派との総合以上の意味を持つ。そこでは，産出＝費用＝所得という，現制度に含意される枠組み（その起源は，Meade, J. E. and Stone, J. R. N., 1969, 344–345）をモデルの数値基礎に据える。会計年度毎の統計上の実際データは，sub-system としての理論的なデータ・セットに修復可能となる。このような最狭義の内生的成長モデル（以下，本モデル）では, 年ごとの経済・財政・金融政策と技術進歩率をはじめとする変数の変動との関係を，すべて，モデル内の数少ない内生パラメータ（*alpha*, *beta*, *delta*, and *lambda*）の変動に結びつけ，それらのパラメータに，R ＆ D・教育・税率等の政策の結果をすべて吸収させている。

2. **本モデルにおける最適と均衡との関係**：国別に, 会計年度毎に，均衡モデルは事後的に成立するが, その均衡には，相当の幅がある。最適均衡は, その究極のあり方である。そのあり方とは，全世界のグローバル化のもとにあっても，一国ごとの持続可能な成長と格差拡大阻止を長期的に達成するようなあり方である。文献上の最適は，ミクロの消費・効用に対する最適をまず指すが，その測定は全体のシステムのなかで，どのように位置づけられ，どのように年度ごとの政策と結びつくのか，数値化は容易ではない。マクロの最適は，会計年度ごとに，事後的均衡のもとに測定でき，ミクロ・ベース方法論の延長・集計とは一線を画すという特性を持つ。本モデルは，マクロの特性（partial なミクロに内在するいくつかの重複をすべて相殺することによって，ミクロとは峻別されるもの）に立脚する。消費の安定的・質的成長こそ，持続的な・結果的な成長の根源的な基盤であると認識している。しかも，消費の質的成長は，内生的な技術進歩率と密接不可分である。成長の結果，格差がおくれながらも是正されるという構造段階は，歴史的には確かに存在したと認識している。いまや，その道筋は，汎国際的な（グローバルを存立させる）平和理念に支えられ，国の発展段階に応じて，個性維持に具体化する。具体化に共通の鍵は，理論的資本・産出比率を早めに，教育・人的資本中心にシフトし，物的資本の膨張から惹起される激しい景気循環を穏やかにすることである。それは，地球社会環境の保全・改善とも，機をいつにする。さらに，本モデルでは，国別には，実物資産を基盤に置き，金融資産は，それに対する補完的・観察的・評価的立場にとどまる。グローバル化には，マーシャルの$k=M/Y$ と実物資産の資本・産出比率 $\Omega=K/Y$ との関係をグローバルに制御する国際的な政策ルールが必要不可欠である。

3. **会計年度ごとの事後的均衡と定常状態均衡との区別**：文献上の定常状態均衡 steady state equilibrium は，所得や資本が一様の増加率で成長する場合であるが，exogenous-oriented の均衡にとどまる。時間に対応する内生的構造式が未形成であったためである。本モデルは，endogenous-oriented の均衡を示す。本モデルの均衡は，会計年度ごとに，事後的に成立する。また，ひとつの会計年度においては，時間（recursive years）ごとに，事前的＝事後的均衡が成立する。内生的構造式があるために，すべてのパラメータと変数がrecursive yearごとに測定される。それを，内生的移行過程 endogenous transitional path という。そ

の過程において，所得や資本が一様の増加率で成長する収束時点は，構造式によって決定される。収束時点は，文献における steady state の場合と同じように存在する。しかし，文献では，時間を表に出さず，資本・労働比率を横軸に用いてその一点に左右から収束するという説明にとどまる。本モデルの収束では，時間 recursive year を支える構造は，資本・労働比率ではなく，資本・産出比率，したがって，資本分配率＝資本・産出比率×資本収益率という式が基礎に置かれる。その理由は，内生的技術進歩率に直接多大な影響を与えるものが資本・産出比率であるためである。国際競争力において，ある国が理論上の資本・産出比率を他の国のそれより相当程度高くすることは，敗北を意味する（資本・産出比率の平準化と上限の存在）。資本分配率は，会計年度ごとに変動するが，移行過程では，一定とされる。もし，資本・産出比率を 1.0 とすると，総要素生産性（total factor productivity）は，資本・労働比率だけで説明できるが，general から遠ざかる。

4.　**政府部門と民間部門との分離：**政府部門の測定の分離は，本モデルと文献上のモデルとの相違を際立たせる。データの集計は，全部門＝政府部門＋民間部門である。実際・所与のソース・データは，前会計年度の国際収支，予算，国の消費と純貯蓄あるいは純投資，政府部門の消費と純貯蓄あるいは純投資，人口とその増減率である。部門別の収益，人件費，資本が，モデル内において，理論値として整合的に測定されるとき，すべてのデータは，実際値を含めて，理論値としてのデータ・セットとなる（Kamiryo Endogenous World Table; KEWT1.07 & 2.08）。現行収支予算制度では，国民経済計算制度（SNA）のもとに，政府部門は，収支差異・財政赤字を計上するにもかかわらず，政府収益は認識されないまま，国富の増減に吸い込まれて，表に出ない。理論的データ・セットでは，政府部門の貯蓄がゼロ以下のマイナスになると，収益も同額のマイナスとなり，収益を差し引いた政府部門産出は，マイナス相当分だけ減少し，産出シェアーは縮小する（極端な財政赤字こそ，デフレの最大要因）。財政赤字の縮小・改善部分は，実際支払税金と理論上の税金との差額の変化によって，数値的に確認可能である。現 EMU のルールは，EU 域内の国が守るべき 3 %（surplus/deficits to GDP）と60%（debts to GDP）を含むが，その理論的構造は，本モデルの政府部門分割によって明示された。緊要な政策とその結果とは，会計年度毎に，公開可能である。EMU ルールに示されたインフレ率抑制も，長期債レート，収束上の成長率と収益率との関係，実際支払賃金率と理論上の賃金率との差異等を用いて，政策と結果に結びつけられる。さらに，政府部門産出＝理論上の租税であることから，格差拡大阻止と結果としての持続可能な成長率との関係を，主要政策セットごとに測定・再検討できる。

5.　**本モデルとコブ・ダグラス生産関数：**コブ・ダグラス生産関数は，規模に対する収穫不変 constant returns to scale（CRS）の制約を持つ。本モデルはなぜコブ・ダグラス生産関数に固執するのか？移行過程において，CRS は，労働と資本との同時的な変動を扱う。そのような CRS は，資本に限定された資本収益率

（diminishing returns to capital, constant returns to capital, および increasing returns to capital（DRC, CRC, and IRC））とどのように整合的に共存できるのか？　本モデルの場合，内生的に測定された資本収益率は，限界資本生産性 marginal productivity of capital に一致する。資本収益率が時間 recursive years とともに減少して，CRC 状態に収束する場合，初期値は，人口成長率がプラスであるかぎり，DRC 状態からスタートする。逆に，資本収益率が時間 recursive years とともに増加する場合，初期値は IRC 状態からスタートする。文献では，IRC 状態は，CRS のもとにおいては，起こり得ない。本モデルでは，人口成長率がマイナスである場合に，IRC 状態を現出するため，CRS の制約を撹拌しない。そのメカニズムは，本モデルがコブ・ダグラス生産関数に隠されている三つのパラメータ（*beta*, *delta*, *lambda*）を発見したことによって成立した。三つのパラメータが構造式のなかに組み込まれたために，CRS という制約をクリアーできたのである。したがって，移行過程においては，内生的に測定される *lambda* を用いて収束スピード・年数を正確に導きだす。収束スピードは，何が緊要な政策であるのかを，会計年度ごとに端的に示唆する。文献上の収束スピード（k からスタートする Sala-i-Martin, Xavier, 1990a, b; Barro, Robert., and Xavier, Sala-i-Martin, 1995；y からスタートする Mankiw Gregory N., David Romer, and David N. Weils, 1992）は，外生的技術進歩率と CRC 仮定のモデル（*delta* の未発見）にとどまるために，会計年度ごとの経済・財政・金融政策の総合的な立案・決定・評価には，直接関与できない。

Appendix: An equation to the speed of convergence

Toshimi Fujumoto[4]

Only skeletons of the Kamiryo model of endogenous economic growth as discussed above in the text are presented here.

Main features of the model

It is obvious that the model depends upon the Cobb-Douglas type production function $Y = BK^{\alpha}K^{1-\alpha}$, where Y, B, K and L are output, technology level, capital input and labor input, respectively.　In order to treat B as of labor-augmenting type and to base the model on the efficient labor basis throughout this appendix for convenience of analysis, we redefine

$B = A^{1-\alpha}$,

4)　I am thankful to the continuous support of Dr. Toshimi Fujimoto in the past. This time, he allowed me to raise his study in this Appendix. The author repeatedly asked him to be the co-author at the earlier paper (JES 10 (Sep, 1): 131-166, but he would not.

$y = Y/AL,$

$k = K/AL,$

$$g_A = \frac{dA/dt}{A},$$

$$g(k) = \frac{dk/dt}{k}$$

so that the model can be represented compactly as

(1) $y = k^{\alpha},$

(2) $\dfrac{dk}{dt} = i_k k^{\alpha} - g_A k - nk,$

(3) $g_A = i_A k^{-(\delta-\alpha)},$

where, as already defined in the text

$$n = \frac{dL}{dt} / L,$$

$$i_K = \frac{\text{saving appropriate to increasing } K}{Y},$$

$$i_A = \frac{\text{saving appropriate to increasing } A}{Y}$$

Thus, the system of nonlinear differential equations (2) (3) determines the dynamics of the model.

Clearly, from (2)

(4) $g(k) = i_K k^{-(1-\alpha)} - (g_A + n)$

is obtained and from the definition $k = K/AL$, it follows that

$$g(k) = g(K) - (g_A + n)$$

so that, comparing this with (4), it is evident that

(5) $g(K) \equiv i_K k^{-(1-\alpha)}.$

Now, let us analyze the structure of the model. To begin with, substituting (3) into (2),

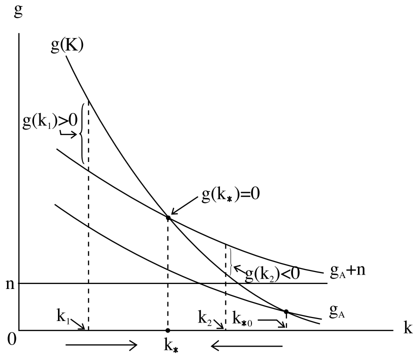
(6) $\dfrac{dk}{dt} = i_K k^{\alpha} - i_A k^{-(\delta-\alpha)} - nk,$

(7) $g(k) = i_K k^{-(1-\alpha)} - i_A k^{-(\delta-\alpha)} - n,$

are obtained. The so-called steady-state value of the pivotal variable k of the model is, as is well-known, nothing but the value, k_*, which makes $\dfrac{dk}{dt} = 0$ in (6). I.e., k_* is the solution of

(8) $i_K k_*^{-(1-\alpha)} - i_A k_*^{-(\delta-\alpha)} - n = 0.$

Diagram



The moving process of k toward k_* can be sketched in the above diagram in case of $\delta > \alpha$. It seems sufficiently apparent at a glance for the process to be stable that the gradient of $g(K)$ be steeper than that of g_A. In this connection, when $\delta < \alpha$, the stability necessarily holds, because g_A then comes under the increasing function of k, and $g(K)$ as a decreasing function of k always cuts g_A from above.

Before proceeding to solve (8), let us mention how to determine the steady-state values of the other endogenous variables, g_A and y. It is almost self-evident that by inserting k_* in (3) and (1), g_{A*} and y_* are determined respectively as follow,

(9) $g_{A*} = i_A k_*^{-(\delta-\alpha)}$,

(10) $y_* = k_*^{\alpha}$.

How to determine k_*

The solution method adopted here is of a kind of linear approximation by way of comparative statics. That is, first of all, put $n = 0$ in (8) to obtain

(11) $k_{*0} = \left(\dfrac{i_K}{i_A}\right)^{\frac{1}{1-\delta}}$.

In short, k_{*0} is the value of k_* in the condition of a constant L.

Secondly, totally differentiate (8) with respect to only (k_*, n) to deduce

$$\left[-(1-\alpha)i_K k_*^{-(1-\alpha)-1} + (\delta-\alpha)i_A k^{-(\delta-\alpha)-1}\right]dk_* = dn$$

and evaluate it at $(n = 0,\ k_* = k_{*0})$, then after rearranging,

(12) $k_{*} = k_{*0}\left[1 + \dfrac{nk_{*0}^{1-\alpha}}{(\delta-\alpha)i_A k_{*0}^{1-\delta} - (1-\alpha)i_K}\right]$

follows, which is found to give what we want to obtain, i.e., k_*. Here, note that

(13) $dk_* = k_* - k_{*0},\ dn = n - 0 = n$

are assumed as a matter of course.

Lastly, substitute (11) in (12) to lead to the final or reduced form of the endogenous variable k_* in the sense of expressing endogenous k_* exclusively in terms of parameters and exogenous variables such as α, δ, i_K, i_A, n. However, the reduced form thus obtained is found too much complicated to deduce any additional meaningful outcomes from it, but substituting (11) in only the denominator of (12) seems to make much contribution to simplify (12) as follows,

(14) $k_* = k_{*0}\left[1 - \dfrac{nk_{*0}^{1-\alpha}}{(1-\delta)i_K}\right]$.

Convergence analysis

First, from the Taylor expansion of (6) at $k = k_*$, a linear approximation

(15) $\dfrac{dk}{dt} \cong \dfrac{\partial}{\partial k}\left(\dfrac{dk}{dt}\right)\Big|_{k=k^*}(k - k^*)$

is obtained. Second, taking (8) (9) into consideration

(16) $\dfrac{\partial}{\partial k}\left(\dfrac{dk}{dt}\right)\Big|_{k=k_*} = \left[(1-\alpha)n + (1+\delta)g_{A^*}\right]$

is found. Now, define for convenience

(17) $\lambda = \left[(1-\alpha)n + (1-\delta)g_{A^*}\right]$

(18) $x = (k - k_*)$

to lead to a differential equation of the simplest type, in place of (15),

(19) $\dfrac{dx}{dt} = -\lambda x$

so that its solution is given as follows, expressing here each time concerned, t,

$$x(t) = x(0)e^{-\lambda t},$$

or more concretely,

(20) $k(t) - k_* = e^{-\left[(1-\alpha)n + (1-\delta)g_{A^*}\right]t}(k(0) - k_*)$.

This is the instrument appropriate for convergence analysis.

(by Toshimi Fujimoto)

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