On the Odd Depth of the Exterior Algebra of Modules

Michinori Sakaguchi

(Received on May 19, 1999)

Schmitt developed the theory of old regular sequences of $\mathbb{Z}_2$-graded rings in [4]. Let $A = A_0 + A_1$ be a $\mathbb{Z}_2$-graded ring and let N be a $\mathbb{Z}_2$-graded A-module. Schmitt defined an odd N-element $\rho \in R_1$ by satisfying the condition that the complex

$$N \xrightarrow{\rho} N \xrightarrow{\rho} N$$

is exact. Let $\rho_1, \rho_2, \ldots, \rho_n$ be a sequence of elements in A_1. Then $\rho_1, \rho_2, \ldots, \rho_n$ is called an odd N-sequence if ρ_i is an odd $\mathrm{N}/\rho_1 \cdots \rho_{i-1}N$-regular element for all $1 \leq i \leq n$. If A is a neotherian $\mathbb{Z}_2$-graded local ring and N is a finite and non-zero graded A-module, then we can consider the maximal length of an odd N-sequence in the same way as the definition of depth N. We call it *odd depth* of N and denote it by odepth N.

Let M be a module over a local ring R. A typical example of $\mathbb{Z}_2$-graded rings is the exterior algebra $E(M)$ of M. If M is a vector space over a field, then the dimension of M is clearly equal to odepth $E(M)$. This result seems to depend on the fact that M is a free module. In general if M has a free direct sumand of rank n, then odd depth of $E(M)$ is greater than n (corollary 1.5). In this paper we study whether the odd depth of $E(M)$ is equal to the greatest number among ranks of free direct summands of M. But we have a simple example in section 3 that there is an odd $E(M)$-regular element $e_1 \in M$ such that the inclusion mapping $Re_1 \to M$ is not split.

We get some conditions in theorem 3.3 to hold this assertion. Namely let

$$0 \longrightarrow K \longrightarrow F = \bigoplus_{i=1}^{t} Ru_i \longrightarrow M \longrightarrow 0$$

be a short exact sequence, where F is a free module. Assume that K is generated by an element over R and odepth $E(M) > 0$. Then there is an element e of M such that Re is a free module and the inclusion mapping $Re \to M$ is a split monomorphism. In particular e is an $E(M)$-regular element. In theorem 3.4 we show that under strict assumptions odepth $E(M)$ is equal to the greatest integer among ranks of free direct summands of M.

In section 1 we state fundamental facts which we need in the following. Let e_1 be an element of M. To get the cohomology of the complex $E(M) \xrightarrow{e_1} E(M) \xrightarrow{e_1} E(M)$, we study the kernel of the mapping $E_i(M) \xrightarrow{e_1} E_{i+1}(M)$ in section 2. We obtain the conditions that e_1 is an odd $E(M)$-regular element in section 3. The odd depth is closely related to the Koszul transform ([3], [4]). Because $E(M)$ is a direct sum of $E_i(M)$, we get another subcomplex of Koszul transforms in section 4. As for $\mathbb{Z}_2$-graded rings we reffer to Chapter 3 in [1].

1. The exterior algebra of modules

Let R be a commutative noetherian local ring with the maximal ideal $\mathfrak{m}$ and the residue field k, and let M be a finite R-module. We write $\mu(M)$ for the minimal number of generators of M. We denote the exterior algebra of the module M by $E(M)$ and also denote the p-th exterior power of M by $E_p(M)$, and note that $E(M) = \oplus_{i\geq 0} E_i(M)$. We use a notation of multiplication $m_1 m_2 \cdots m_t$ on $E(M)$ in stead of the wedge notation $m_1 \wedge m_2 \wedge \cdots \wedge m_t$ which is very commonly used in the theory of exterior algebras.

Let A be the exterior algebra $E(M)$. Set

$$A_0 = \bigoplus_{i\geq 0} E_{2i}(M), \quad A_1 = \bigoplus_{i\geq 0} E_{2i+1}(M),$$

and

$$\mathfrak{M}_0 = \mathfrak{m} \oplus (\bigoplus_{i\geq 1} E_{2i}(M)), \quad \mathfrak{M}_1 = A_1 \textit{ and } \mathfrak{M} = \mathfrak{M}_0 + \mathfrak{M}_1,$$

We write $\Phi = (\mathfrak{M} / \mathfrak{M}^2)_1$ as 2.6 in [4].

Proposition 1.1. *With notation as above, $A = A_0 + A_1$ is a local $\mathbb{Z}_2$-graded ring and $\mathfrak{M}$ is the maximal ideal of A with $A/\mathfrak{M} = k$. Furthermore we have $\Phi = M/\mathfrak{m}M$, in particular* $\dim_k \Phi = \mu(M)$.

Proof. As it is clear except the fact that $\Phi = M/\mathfrak{m}M$, we only prove its equality. Because $E_i(M)\,E_j(M) = E_{i+j}(M)$ for all $i, j \geqq 0$, we see that

$$\begin{aligned}\mathfrak{M}_0\mathfrak{M}_1 &= (m \oplus E_2(M) \oplus E_4(M) \oplus \cdots)(E_1(M) \oplus E_3(M) \oplus E_5(M) \oplus \cdots) \\ &= mE_1(M) \oplus (mE_3(M) + E_2(M)E_1(M)) \\ &\quad \oplus (mE_5(M) + E_2(M)E_3(M) + E_4(M)E_1(M)) \oplus \cdots \\ &= mE_1(M) \oplus E_3(M) \oplus E_5(M) \oplus \cdots\end{aligned}$$

Therefore, since $\mathfrak{M}/\mathfrak{M}^2 = \left(\mathfrak{M}_0/(\mathfrak{M}_0^2+\mathfrak{M}_1^2)\right) \oplus (\mathfrak{M}_1/\mathfrak{M}_0\mathfrak{M}_1)$, we have that

$$\Phi = \mathfrak{M}_1/\mathfrak{M}_0\mathfrak{M}_1 = (\bigoplus_{i\geq 0} E_{2i+1})/\left(\mathfrak{m}E_1(M) \oplus \left(\bigoplus_{i\geq 1} E_{2i+1}(M)\right)\right)$$

$$= E_1(M)/mE_1(M) = M/mM,$$

which proves our proposition.

If m is an element of M, then m is an element of A_1. We easily see next lemma by definition.

Lemma 1.2. *Let $A = E(M)$ and m be an element of M. Then m is A-regular if and only if the sequence*

$$0 \longrightarrow R \xrightarrow{\ m\ } E_1(M) \xrightarrow{\ m\ } E_2(M) \xrightarrow{\ m\ } E_3(M) \xrightarrow{\ m\ } \cdots$$

is exact.

Lemma 1.3. *Assume thet $M = Rm$ for some $m \in M$. Then the following are equivalent:*

1. *m is $E(M)$-regular.*
2. *The sequence $0 \longrightarrow R \xrightarrow{\ m\ } M \longrightarrow 0$ is exact.*
3. *M is free of rank one.*

Proof. This lemma follows from the fact that $E_2(M) = 0$.

Theorem 1.4. (Theorem 8 of chapter 5 in [2]). *Let $M_1, M_2, \ldots, M_q$ be R-modules. Then $E(M_1 \oplus M_2 \oplus \cdots \oplus M_q)$ and $E(M_1) \otimes_R E(M_2) \otimes_R \cdots \otimes_R E(M_q)$ are isomorphic $\mathbb{Z}_2$-graded rings under an isomorphism which matches $m_1 + m_2 + \cdots + m_q$ with $(m_1 \otimes 1 \otimes \cdots \otimes 1) + (1 \otimes m_2 \otimes \cdots \otimes 1) + \cdots + (1 \otimes 1 \otimes \cdots \otimes m_q)$.*

Corollary 1.5. *Let M be a non-zero, finite R-module and F a free R-submodule of M. Suppose that $M = F \oplus N$ with some non-zero submodule N of M. Then we have* odepth $E(M)$ = rank F + odepth $E(N)$.

Proof. At first assume that F is of rank 1 and $F = Re$. Then, by theorem, there is an isomorphism $\phi: E(M) \to E(F) \otimes_R E(N)$ with $\phi(e) = e \otimes 1$. Since e is a free base of F, the natural mapping $R \to Re$ is an isomorphism, whence the mapping $R \otimes_R E(N) \to Re \otimes_R E(N)$ is also so. Because

$$E(F)\otimes_R E(N)=\big(R\otimes_R E(N)\big)\oplus\big(Re\otimes_R E(N)\big)\big),$$

the sequence

$$E(M)\xrightarrow{\ e\ }E(M)\xrightarrow{\ e\ }E(M)$$

is exact and $E(M)/eE(M) = N$. Therefore we see that odepth $E(M)$ = 1 + odepth $E(N)$ = rank F + odepth $E(N)$.

We can prove easily our corollary by induction on rank F.

2. Annihilators of an element of M

We denote the module of column vectors of size n by R^n. Let $\boldsymbol{a}$, $\boldsymbol{b}$ be elements of R^n. Then we denote the inner product of $\boldsymbol{a}$ and $\boldsymbol{b}$ by $(\boldsymbol{a}, \boldsymbol{b})$. Let M be a non-zero R-module with $\mu(M) = t$ and let $e_1, \dots, e_t$ be a minimal basis of M over R. Let $F = \bigoplus_{i=1}^{t} Ru_i$ be a free R-module of rank t. Then we have a natural exact sequence

$$0\longrightarrow K\longrightarrow F\xrightarrow{\ \varphi\ }M\longrightarrow 0$$

with $\varphi(u_i) = e_i$. Let $f_1, \dots, f_l$ be a set of generators of K over R and assume that

$$f_i = s_{i1}u_1+\cdots+s_{it}u_t \quad 1\le i\le l$$

with $s_{ij} \in \mathrm{R}$. We write $\boldsymbol{s}_j = {}^t(s_{1j}, \cdots, s_{lj})$.

Proposition 2.1. For $r \in R$, $re_1 = 0$ *if and only if there exists an element* $\boldsymbol{c}$ *of* R^l *such that* $r = (\boldsymbol{c}, \boldsymbol{s}_1)$ and $(\boldsymbol{c}, \boldsymbol{s}_j) = 0$ for all $j \ge 2$.

Proof. Assume that $re_1 = 0$. Then there exist elements c_i $(1 \le i \le l)$ in R such that

$$ru_1=\sum_i c_i f_i=\sum_j\Big(\sum_i c_i s_{ij}\Big)u_j.$$

Let $\boldsymbol{c} = {}^t(c_1, \dots, c_l)$. Then $ru_1 = \sum_j(\boldsymbol{c}, \boldsymbol{s}_j)u_j$ and therefore comparing coeffcients of u_i, we see that $r = (\boldsymbol{c}, \boldsymbol{s}_1)$ and $(\boldsymbol{c}, \boldsymbol{s}_j) = 0$ for all $2 \le j \le t$.

Conversely suppose that $r = (\boldsymbol{c}, \boldsymbol{s}_1)$ and $(\boldsymbol{c}, \boldsymbol{s}_j) = 0$ for all $j \ge 2$. Because $\sum_j s_{ij}e_j = 0$ for all i, we see that

$$re_1=\sum_{j=1}^{t}(\boldsymbol{c},\boldsymbol{s}_j)e_j=\sum_{i=1}^{\ell}c_i\sum_{j=1}^{t}s_{ij}e_j=0,$$

and the proof is complete.

Let p be an integer with $1 \le p \le t-1$. Then $u_{j_1}u_{j_2}\cdots u_{j_p}$ $(1 \le j_1 < j_2 < \cdots < j_p \le t)$ is a free besis of $E_p(F)$ over R. By theorem 4 of chapter 5 in [2], we see that $E_p(M)$ is isomorphic to $E_p(F)/KE_{p-1}(F)$, and hence $e_{j_1}e_{j_2}\cdots e_{j_p}$ $(1 \le j_1 < j_2 < \cdots < j_p \le t)$ is a set of generators of $E_p(M)$ over R. We have a diagram

$$\begin{array}{ccc} E_p(F) & \xrightarrow{u_1} & E_{p+1}(F) \\ \downarrow & & \downarrow \\ E_p(M) & \xrightarrow{e_1} & E_{p+1}(M). \end{array}$$

Proposition 2.2. *With notation as above, let g be an element of $E_p(M)$ with $1 \le p \le t-1$. Then $e_1 g = 0$ if and only if there exist elements $\boldsymbol{c}_{j_1 j_2 \cdots j_p}$ $(2 \le j_1 < j_2 < \cdots < j_p \le t)$ of R^l such that the following conditions hold:*

1. *g is of the form that*

$$g = e_1 h + (-1)^p \sum_{2 \le j_1 < j_2 < \cdots < j_p} (\boldsymbol{c}_{j_1 j_2 \cdots j_p}, \boldsymbol{s}_1) e_{j_1} e_{j_2} \cdots e_{j_p}$$

with some $h \in E_{p-1}(M)$.

2. *For all $2 \le j_1 < \cdots < j_k < \cdots < j_{p+1} \le t$, we have*

$$\sum_{k=1}^{p+1} (-1)^{p+1-k} (\boldsymbol{c}_{j_1 \cdots \hat{j}_k \cdots j_{p+1}}, \boldsymbol{s}_{j_k}) = 0$$

where the ^ over j_k indicates that this factor is to be omitted.

Proof. Assume that $e_1 g = 0$. Since $g \in E_p(M)$, we can write

$$g = e_1 h + \sum_{2 \le j_1 < \cdots < j_p} a_{j_1 \cdots j_p} e_{j_1} \cdots e_{j_p}$$

with $a_{j_1 \cdots j_p} \in R$ and some $h \in E_{p-1}(M)$. To prove our proposition we may also suppose that

$$g = \sum_{2 \le j_1 < \cdots < j_p} a_{j_1 \cdots j_p} e_{j_1} \cdots e_{j_p}$$

because $e_1^2 = 0$. By the above diagram there are elements $c_{ij_1 \cdots j_p}$ of R such that

$$u_1 \sum_{2 \le j_1 < \cdots < j_p} a_{j_1 \cdots j_p} u_{j_1} \cdots u_{j_p} = \sum_{i=1}^{l} \left(\sum_{1 \le j_1 < \cdots < j_p} c_{ij_1 \cdots j_p} u_{j_1} \cdots u_{j_p} \right) \left(\sum_{k=1}^{t} s_{ik} u_k \right)$$

We put

$$\boldsymbol{c}_{j_1 j_2 \cdots j_p} = {}^t(c_{1 j_1 \cdots j_p}, \cdots, c_{l j_1 \cdots j_p})$$

for $1 \le j_1 < \cdots < j_p$. Comparing the coefficient of $u_1 u_{j_1} \cdots u_{j_p}$ with $j_1 \ge 2$, we see that

$$a_{j_1 \cdots j_p} = (-1)^p (\boldsymbol{c}_{j_1 \cdots j_p}, \boldsymbol{s}_1) + (-1)^{p-1} (\boldsymbol{c}_{1 \hat{j}_1 \cdots j_p}, \boldsymbol{s}_{j_1}) + \cdots + (-1)^{p-p} (\boldsymbol{c}_{1 j_1 \cdots \hat{j}_p}, \boldsymbol{s}_{j_p}).$$

It follows from the coefficient of $u_{j_1} \cdots u_{j_{p+1}}$ with $2 \le j_1 < \cdots < j_{p+1} \le t$ that

$$(\boldsymbol{c}_{j_1\cdots j_p\hat{j}_{p+1}},\ \boldsymbol{s}_{j_{p+1}})+(-1)(\boldsymbol{c}_{j_1\cdots\hat{j}_p j_{p+1}},\ \boldsymbol{s}_{j_p})+\cdots+(-1)^p(\boldsymbol{c}_{\hat{j}_1 j_2\cdots j_{p+1}},\ \boldsymbol{s}_{j_1})=0,$$

which shows the condition 2.

Now g can be expressed as

$$g=(-1)^p\sum_{2\le j_1<\cdots<j_p}(\boldsymbol{c}_{j_1\cdots j_p},\ \boldsymbol{s}_1)\,e_{j_1}\cdots e_{j_p}.$$

$$+\sum_{2\le j_1<\cdots<j_p}\Big((-1)^{p-1}(\boldsymbol{c}_{1\hat{j}_1\cdots j_p},\ \boldsymbol{s}_{j_1})+\cdots+(-1)^{p-p}(\boldsymbol{c}_{1j_1\cdots\hat{j}_p},\ \boldsymbol{s}_{j_p})\Big)\,e_{j_1}\cdots e_{j_p}.$$

We set

$$g_1=\sum_{2\le j_1<\cdots<j_p}\Big((-1)^{p-1}(\boldsymbol{c}_{1\hat{j}_1\cdots j_p},\ \boldsymbol{s}_{j_1})+\cdots+(-1)^{p-p}(\boldsymbol{c}_{1j_1\cdots\hat{j}_p},\ \boldsymbol{s}_{j_p})\Big)\,e_{j_1}\cdots e_{j_p}.$$

Then it is sufficent to prove the part of "only if" that we show $g_1\in e_1E_{p-1}(M)$. We fix a sequence $2\le j_2<j_3<\cdots<j_p\le t$ and we consider the following partial sum $g_{j_2\cdots j_p}$ of the terms of g_1

$$\begin{aligned}
g_{j_2\cdots j_p}:=&\sum_{2\le j_1<j_2}(-1)^{p-1}(\boldsymbol{c}_{1\hat{j}_1 j_2\cdots j_p},\ \boldsymbol{s}_{j_1})\,e_{j_1}e_{j_2}\cdots e_{j_p}\\
&+\ \cdots\cdots\cdots\cdots\\
&+\sum_{j_k<j_1<j_{k+1}}(-1)^{p-k}(\boldsymbol{c}_{1j_2\cdots j_k\hat{j}_1 j_{k+1}\cdots j_p},\ \boldsymbol{s}_{j_1})\,e_{j_2}\cdots e_{j_k}e_{j_1}e_{j_{k+1}}\cdots e_{j_p}\\
&+\ \cdots\cdots\cdots\cdots\\
&+\sum_{j_p<j_1\le t}(-1)^{p-p}(\boldsymbol{c}_{1j_2\cdots j_p\hat{j}_1},\ \boldsymbol{s}_{j_1})\,e_{j_2}\cdots e_{j_{p-1}}e_{j_1}\\
=&\sum_{2\le j_1<j_2}(-1)^{p-1}(\boldsymbol{c}_{1j_2\cdots j_p},\ \boldsymbol{s}_{j_1})e_{j_1}e_{j_2}\cdots e_{j_p}\\
&+\ \cdots\cdots\cdots\cdots\\
&+\sum_{j_k<j_1<j_{k+1}}(-1)^{p-1}(\boldsymbol{c}_{1j_2\cdots j_p},\ \boldsymbol{s}_{j_1})\,e_{j_1}e_{j_2}\cdots e_{j_k}e_{j_{k+1}}\cdots e_{j_p}\\
&+\ \cdots\cdots\cdots\cdots\\
&+\sum_{j_p<j_1\le t}(-1)^{p-1}(\boldsymbol{c}_{1j_2\cdots j_p},\boldsymbol{s}_{j_1})\,e_{j_1}e_{j_2}\cdots e_{j_{p-1}}.
\end{aligned}$$

Since $\sum_{j_1=1}^{t}s_{ij_1}e_{j_1}=0$ for all $1\le i\le l$, we see that $\sum_{j_1=1}^{t}(\boldsymbol{c}_{1j_2\cdots j_p},\ \boldsymbol{s}_{j_1})\,e_{j_1}=0$, and hence $\sum_{j_1=1}^{t}(\boldsymbol{c}_{1j_2\cdots j_p},\ \boldsymbol{s}_{j_1})\,e_{j_1}e_{j_2}\cdots e_{j_p}=0$. Consequently we get that

$$(-1)^{p-1}(\boldsymbol{c}_{1j_2\cdots j_p},\ \boldsymbol{s}_1)\,e_1e_{j_2}\cdots e_{j_p}+(-1)^{p-1}\sum_{\substack{j_1=2\\ j_1\ne j_2,\ldots,j_p}}^{t}(\boldsymbol{c}_{1j_2\cdots j_p},\ \boldsymbol{s}_{j_1})\,e_{j_1}e_{j_2}\cdots e_{j_p}=0$$

because $e_{j_k}^2=0$. Thus $g_{j_2\cdots j_p}=(-1)^p(\boldsymbol{c}_{1j_2\cdots j_p},\boldsymbol{s}_1)e_1e_{j_2}\cdots e_{j_p}\in e_1E_{p-1}(M)$, and therefore we obtain that $g_1=\sum_{2\le j_2<\cdots<j_p}g_{j_2\cdots j_p}\in e_1E_{p-1}(M)$.

Next we shall prove its converse. Assume that

$$g = e_1 h + (-1)^p \sum_{2 \le j_1 < j_2 < \cdots j_p} (\boldsymbol{c}_{j_1 j_2 \cdots j_p}, \boldsymbol{s}_1) e_{j_1} e_{j_2} \cdots e_{j_p}$$

and for all $2 \le j_1 < \cdots < j_k < \cdots < j_{p+1} \le t$, we have

$$\sum_{k=1}^{p+1} (-1)^{p+1-k} (\boldsymbol{c}_{j_1 \cdots \hat{j}_k \cdots j_{p+1}}, \boldsymbol{s}_{j_k}) = 0.$$

Let $c_{ij_1 j_2 \cdots j_p}$ be elements of R with $\boldsymbol{c}_{j_1 j_2 \cdots j_p} = {}^t(c_{1j_1 \cdots j_p}, \dots, c_{lj_1 \cdots j_p})$. Since $\sum_k s_{ik} e_k = 0$ $(1 \le i \le l)$, we get that

$$\sum_{i=1}^{l} \Big(\sum_{2 \le j_1 < \cdots < j_p} c_{ij_1 \cdots j_p} e_{j_1} \cdots e_{j_p} \Big) \Big(\sum_{k=1}^{t} s_{ik} e_k \Big) = \sum_{2 \le j_1 < \cdots < j_p} \sum_{k=1}^{t} (\boldsymbol{c}_{j_1 \cdots j_p}, \boldsymbol{s}_k) e_{j_i} \cdots e_{j_p} e_k = 0,$$

and hence

$$\sum_{2 \le j_1 < \cdots < j_p} (\boldsymbol{c}_{j_1 \cdots j_p}, \boldsymbol{s}_1) e_{j_1} \cdots e_{j_p} e_1 + \sum_{2 \le j_1 < \cdots < j_p} \sum_{k=2}^{t} (\boldsymbol{c}_{j_1 \cdots j_p}, \boldsymbol{s}_k) e_{j_1} \cdots e_{j_p} e_k = 0.$$

As the previous method we put

$$g'_{j_1 \cdots j_{p+1}} = \sum_{k=1}^{p+1} (\boldsymbol{c}_{j_1 \cdots \hat{j}_k \cdots j_{p+1}}, \boldsymbol{s}_{j_k}) e_{j_1} \cdots \hat{e}_{j_k} \cdots e_{j_{p+1}} e_{j_k}$$

for $2 \le j_1 < \cdots < j_{p+1} \le t$. They by assumption we see that

$$g'_{j_1 \cdots j_{p+1}} = \left(\sum_{k=1}^{p+1} (-1)^{p+1-k} (\boldsymbol{c}_{j_1 \cdots \hat{j}_k \cdots j_{p+1}}, \boldsymbol{s}_{j_k}) \right) e_{j_1} e_{j_2} \cdots e_{j_{p+1}} = 0.$$

If $k = j_q$ for some $1 \le q \le p$, then $e_{j_1} \cdots e_{j_p} e_k = 0$. Whence this implies that

$$\sum_{2 \le j_1 < \cdots < j_p} \sum_{k=2}^{t} (\boldsymbol{c}_{j_1 \cdots j_p}, \boldsymbol{s}_k) e_{j_1} \cdots e_{j_p} e_k = \sum_{2 \le j_1 < \cdots < j_{p+1}} g'_{j_1 \cdots j_{p+1}} = 0,$$

and therefore

$$\sum_{2 \le j_1 < \cdots < j_p} (\boldsymbol{c}_{j_1 \cdots j_p}, \boldsymbol{s}_1) e_{j_1} \cdots e_{j_p} e_1 = 0.$$

Thus it follows that

$$e_1 g = (-1)^p \sum_{2 \le j_1 < \cdots < j_p} (\boldsymbol{c}_{j_1 \cdots j_p}, \boldsymbol{s}_1) e_1 e_{j_1} \cdots e_{j_p} = 0,$$

and this completes our proof.

Corollary 2.3. *Notation being as proposition* 2.2, *we have the following*:

1. *For* $g \in M$, $e_1 g = 0$ *if and only if there exist an element* a *of* R *and elements* $\boldsymbol{c}_2, \dots, \boldsymbol{c}_t$ of R^l *such that* $g = ae_1 + \sum_{i=2}^{t} (\boldsymbol{c}_i, \boldsymbol{s}_1) e_i$ *and* $(\boldsymbol{c}_i, \boldsymbol{s}_j) = (\boldsymbol{c}_j, \boldsymbol{s}_i)$ *for all* $i, j \ge 2$.

2. *Let* $\mathfrak{a}$ *be an ideal in R generated by* $s_{11}, s_{21}, \ldots, s_{l1}$ *and assume that* $t \geq 2$. *For* $g \in E_{t-1}(M)$, $e_1 g = 0$ *if and only if there exist an element a of* $\mathfrak{a}$ *and an element h of* $E_{t-2}(M)$ *such that* $g = e_1 h + a e_2 \cdots e_t$.

Proof. 1. Replacing $\boldsymbol{c}_i$ by $-\boldsymbol{c}_i$, this statement directly comes from proposition 2.2.

2. Let $p = t - 1$. Then, by proposition 2.2, $e_1 g = 0$ if and only if

$$g = e_1 h + (-1)^{t-1}(\boldsymbol{c}_{2\cdots t}, \boldsymbol{s}_1) e_2 \cdots e_t.$$

We put $a = (-1)^{t-1}(\boldsymbol{c}_{2\cdots t}, \boldsymbol{s}_1)$. Then $a \in \mathfrak{a}$ and it completes the proof.

3. An odd regular element

We retain the notation introduced in the previous section. In the following proposition we shall show a condition equivalent to that Re_1 is a free direct summand of M.

Proposition 3.1. *Let the assumption be as in proposition* 2.2 *and assume that* $t = \mu(M) \geq 2$. *Then the following conditions are equivalent.*

1. Re_1 *is a free module of rank one and the inclusion mapping* $Re_1 \to M$ *is a split monomorphism.*
2. $\boldsymbol{s}_1$ *is an R-linear combination of* $\boldsymbol{s}_2, \ldots, \boldsymbol{s}_t$.

Proof. Suppose that Re_1 is free and the mapping $Re_1 \to M$ is split. Then there exists a homomorphism $\varphi: M \to Re_1$ such that $\varphi(e_1) = e_1$. Put $\varphi(e_i) = r_i e_1$ $(1 \leq i \leq t)$, where $r_1 = 1$. Since $\sum_j s_{ij} e_j = 0$, it follows that $\varphi(\sum_j s_{ij} e_j) = \sum_j r_j s_{ij} e_1 = 0$ for all $1 \leq i \leq l$. Therefore, because Re_1 is free, we see that $\sum_j r_j \boldsymbol{s}_j = \boldsymbol{0}$, and this shows that $\boldsymbol{s}_1$ is an R-linear combination of $\boldsymbol{s}_2 \ldots, \boldsymbol{s}_t$.

Next assume that $\boldsymbol{s}_1 + \sum_{j=2}^{t} r_j \boldsymbol{s}_j = \boldsymbol{0}$ with $r_j \in R$. We let $\psi: F = \bigoplus_{i=1}^{t} Ru_i \to Re_1$ be a homomorphism such that $\psi(u_1) = e_1$ and $\psi(u_1) = r_i e_1$ $(2 \leq i \leq t)$. Since $\psi(f_i) = \psi(\sum_j s_{ij} u_j) = 0$ for all i, we get a homomorhpism $\varphi: M \to Re_1$ such that $\varphi(e_1) = e_1$, and it implies that $Re_1 \to M$ is split. By proposition 2.1 if $re_1 = 0$, then there is an element $\boldsymbol{c}$ of R^l such that $r = -(\boldsymbol{c}, \boldsymbol{s}_1)$ and $(\boldsymbol{c}, \boldsymbol{s}_i) = 0$ for all $i \geq 2$. Then by assumption we know that $r = 0$ and hence Re_1 is a free module of rank one.

Proposition 3.2. *Under the assumption in proposition* 2.2 *we set* $t = \mu(M)$. *Then* e_1 *is an odd*

$E(M)$-regular element if and only if the following condition hold:

1. *If $\boldsymbol{c}$ is an element of R^l such that $(\boldsymbol{c}, \boldsymbol{s}_i) = 0$ for all $i \geq 2$, then $(\boldsymbol{c}, \boldsymbol{s}_1) = 0$.*
2. *For each p $(1 \leq p \leq t-1)$, if $\boldsymbol{c}_{j_1 j_2 \cdots j_p}$ $(2 \leq j_1 < j_2 < \cdots < j_p \leq t)$ are elements of R^l such that*

$$\sum_{k=1}^{p+1} (-1)^{p+1-k} (\boldsymbol{c}_{j_1 \cdots \hat{j}_k \cdots j_{p+1}}, \boldsymbol{s}_{j_k}) = 0$$

for all $2 \leq j_1 < \ldots < j_k < \ldots < j_{p+1} \leq t$, then we have that

$$(-1)^p \sum_{2 \leq j_1 < j_2 < \cdots < j_p} (\boldsymbol{c}_{j_1 j_2 \cdots j_p}, \boldsymbol{s}_1) e_{j_1} e_{j_2} \cdots e_{j_p} \in e_1 E_{p-1}(M).$$

Proof. By lemma 1.2, e_1 is $E(M)$-regular if and only if

$$0 \longrightarrow R \xrightarrow{e_1} E_1(M)$$

is exact and for all p $(1 \leq p \leq \mu(M) - 1)$

$$E_{p-1}(M) \xrightarrow{e_1} E_p(M) \xrightarrow{e_1} E_{p+1}(M)$$

is exact. Then this proposition follows from proposition 2.1 and 2.2.

By corollay 1.5 and proposition 3.1, we see that if $\boldsymbol{s}_1$ is an R-linear combination of $\boldsymbol{s}_2, \ldots, \boldsymbol{s}_t$, then the conditions of proposition 3.2 hold.

Theorem 3.3. *Let R be a local ring and let M be a finite R-module with $t = \mu(M) \geq 2$. Let*

$$0 \longrightarrow K \longrightarrow F = \bigoplus_{i=1}^{t} Ru_i \longrightarrow M \longrightarrow 0$$

be a short exact sequence, where F is a free module. Assume that K is generated by an element over R and odepth $E(M) > 0$. *Then there is an element e of M such that Re is a free module of rank one and the inclusion mapping $Re \rightarrow M$ is a split monomorphism. In particular e is an $E(M)$-regular element.*

Proof. We put $A = E(M)$. Since odepth $A > 0$, there is an A-regular element $\rho \in A_1$. Because $A_1 = E_1(M) \oplus (\oplus_{i \geq 1} E_{2i+1}(M))$ and each element in $E_{2i+1}(M)$ is of the form $\sum b_i \tau_i$ with $b_i \in A_0$ and $\tau_i \in A_1$, we can write $\rho = \sum a_i e_i + \sum b_i \tau_i$, where $a_1 \in R$ and $e_1, \ldots, e_t$ is a set of minimal generators of M. It follows from lemma 1 in [2] that $\sum a_i e_i$ is A-regular, whence we may assume that $\rho = \sum a_i e_i$. Furthermore by theorem at 2.6 in [2], we see that $a_i \notin \mathfrak{m}$ for some i. Therefore replacing $e_1, \ldots, e_t$ we may also assume that $\rho = e_1$ and the homomorphism $F \to M$ sends u_i to e_i.

By lemma 1.2, the sequence

$$E_{t-2}(M) \xrightarrow{e_1} E_{t-1}(M) \xrightarrow{e_1} E_t(M)$$

is exact. Let f generate K and put

$$f = s_1u_1 + s_2u_2 + \cdots + s_tu_t.$$

Then, by proposition 3.2, for any $c_{23\cdots t} \in R$,

$$c_{23\cdots t}s_1e_2e_3\cdots e_t \in e_1E_{t-2}(M),$$

in particular $s_1e_2e_3\cdots e_t \in e_1E_{t-2}(M)$. Thus there are element a_i $(2 \le i \le t)$ and b_{ij} $(1 \le i < j \le t)$ of R such that

$$s_1u_2\cdots u_t = \sum_{i\ge 2} a_1u_1u_2\cdots\hat{u}_i\cdots u_t + (s_1u_1 + s_2u_2 + \cdots + s_tu_t)(\sum_{i<j} b_{ij}u_1\cdots\hat{u}_i\cdots\hat{u}_j\cdots u_t).$$

Comparing the coefficient of $u_2u_3\cdots u_t$ we get that

$$s_1 = b_{12}s_2 + (-1)b_{13}s_3\cdots + (-1)^{t-2}b_{1t}s_t$$

and hence if follows from proposition 3.1 that Re_1 is free and the inclusion mapping $Re_1 \to M$ is split. This completes the proof.

Example. Let k be a field and X, Y variables over k. Consider a local ring $R = k[X, Y]/(X, Y)^2 = k[x, y]$ with maximal ideal $\mathfrak{m} = (x, y)$. We set $s_{12} = x$, $s_{22} = y$ and

$$\begin{pmatrix} s_{11} \\ s_{21} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} s_{12} \\ s_{22} \end{pmatrix},$$

where all a_{ij} are elements of R. But, since $\mathfrak{m}^2 = 0$, we may assume $a_{ij} \in k$. Let $Ru_1 \oplus Ru_2$ be a free R-module and put

$$M = \frac{Ru_1 \oplus Ru_2}{R(s_{11}u_1 + s_{12}u_2) + R(s_{21}u_1 + s_{22}u_2)} = Re_1 + Re_2.$$

We shall prove that e_1 is $E(M)$-regular. It is sufficient to show that the conditions of proposition 3.2 hold. First let $\boldsymbol{c} = {}^t(c_1, c_2)$ be an element of R^2 such that $(\boldsymbol{c}, \boldsymbol{s}_2) = 0$. Then $c_1x + c_2y = 0$, and thus $c_1, c_2 \in m$. Therefore $(\boldsymbol{c}, \boldsymbol{s}_1) = 0$, because $s_{11}, s_{21} \in m$. We next observe that $(\boldsymbol{c}_2, \boldsymbol{s}_1)e_2 \in Re_1$ for all $\boldsymbol{c}_2 \in R^2$. Clearly it suffices to show that $s_{i1}e_2 \in Re_1$ for $i = 1, 2$. It follows from the equations

$$s_{11}e_2 = -(a_{11}s_{11} + a_{12}s_{21})e_1,\quad s_{21}e_2 = -(a_{21}s_{11} + a_{22}s_{21})e_1$$

and consequently e_1 is $E(M)$-regular.

However Re_1 is not always a free direct summand of M. By proposition 3.1 we shall seek a condition equivalent to that $\boldsymbol{s}_1 = b\boldsymbol{s}_2$ for some b in R. Now as previous way we may assume

$s_1 = bs_2$ for some b in k and we set

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then it follows that $As_2 = bs_2$ and so $(A - bE)s_2 = \mathbf{0}$. Finally we obtain that Re_1 is a free direct summand of M if and only if $A = bE$.

Theorem 3.4. *Let R be a local normal domain and let M be a finite R-module with $\mu(M) = 2$. Then* odepth $E(M) > 0$ *if and only if there is an element e_l of M such that Re_1 is free and the inclusion mapping $Re_1 \to M$ is a split monomorphism.*

Proof. One direction follows from corollary 1.5, so we assume that odepth $E(M) > 0$. Let

$$0 \longrightarrow K \longrightarrow F = \bigoplus_{i-1}^{2} Ru_i \longrightarrow M \longrightarrow 0$$

be a short exact sequence, where F is a free moduke. Let e_i be the image of u_i. As the same method in the proof of theorem 3.3 we may suppose that e_1 is $E(M)$-regular. Then the complex

$$0 \longrightarrow R \xrightarrow{e_1} M \xrightarrow{e_1} E_2(M) \longrightarrow 0$$

is exact. Let K be generated by $f_i = s_{i1}u_1 + s_{i2}u_2 (1 \le i \le l)$ over R. Set $\mathbf{s}_1 = {}^t(s_{11}\ s_{21}\ \cdots\ s_{l1})$, $\mathbf{s}_2 = {}^t(s_{12}\ s_{22}\ \cdots\ s_{l2})$ and $\mathfrak{a} = \sum Rs_{i1}$. Then by corollary 2.3 we see that e_1 is $E(M)$-regular if and only $\mathrm{Ann}_R e_1 = 0$ and $\mathfrak{a}e_2 \subset Re_1$, where we denote by $\mathrm{Ann}_R e_1$ the set of elements $a \in R$ such that $ae_1 = 0$. Because our assertion is clear in the case $\mathbf{s}_1 = \mathbf{0}$, we may assume $\mathbf{s}_1 \neq \mathbf{0}$.

We claim that if $(\mathbf{c}, \mathbf{s}_2) = 0$ for $\mathbf{c} = {}^t(c_1\ c_2\ \cdots\ c_t) \in R^l$, then $(\mathbf{c}, \mathbf{s}_1) = 0$. In fact assume $(\mathbf{c}, \mathbf{s}_2) = 0$. Then $\sum c_i(s_{i1}e_1 + s_{i2}e_2) = 0$ and so $(\mathbf{c}, \mathbf{s}_1)e_1 = 0$, hence $(\mathbf{c}, \mathbf{s}_1) = 0$. Let K be quotient field of R. Then we can write $\mathbf{s}_1 = \alpha \mathbf{s}_2$ with some $\alpha \in K$. On the other hand since $\mathfrak{a}e_2 \subset Re_1$, there is an element a_i such that $s_{i1}e_2 = a_ie_1$. Thus we can find elements d_{ij} of R such that $a_iu_1 - s_{i1}u_2 = \sum d_{ij}f_j$. It therefore follows that $s_{i1} = \sum -d_{ij}s_{j2}$. We put a matrix $C = (-d_{ij})$. Then we obtain $\mathbf{s}_1 = C\mathbf{s}_2$ and hence $(C - \alpha E)\mathbf{s}_2 = \mathbf{0}$. This impies $|C - \alpha E| = 0$ because $\mathbf{s}_2 \neq \mathbf{0}$. Since R is a normal domain, $\alpha \in R$. By proposition 3.1 Re_1 is free and the inclusion mapping $Re_1 \to$ M is a split monomorphism. We prove our theorem.

4. Koszul tramsforms of exterior algebras

Let R be a local ring, $\mathfrak{m}$ the maximal ideal of R and let M be a non-zero finite R-module. We set $A = E(M) = \oplus_{i\geq 0} E_i$, and $E_i = 0$ for $i < 0$. Let $e_1, \ldots, e_t$ be a set of minimal generators of M and let $r_1, \ldots, r_s$ be minimal generators of $\mathfrak{m}$. Let $A[\xi_1, \ldots, \xi_s | x_1, \ldots, x_t] = A[\xi|x]$ be a polynomial ring with odd variables ξ_i and even variables x_j. Then $A[\xi|x]$ is a $\mathbb{Z}_2$-graded ring. Let $\delta = \sum_{i=1}^{s} r_i\xi_i + \sum_{j=1}^{t} e_j x_j$. Then $\delta \in A[\xi|x]_1$ and we have a complex

$$A[\xi|x] \xrightarrow{\delta} A[\xi|x] \xrightarrow{\delta} A[\xi|x].$$

The cohomology of which is called the *Koszul transform* of A and is denoted by $\mathfrak{C}_A$ (cf. 4.1 in [2]). Let $A[\xi|x]_{[n]}$ be the set of homogeneous polynomials of degree n. Then the above complex is as follow

$$c:\quad 0 \longrightarrow A \xrightarrow{\delta} A[\xi|x]_{[1]} \xrightarrow{\delta} \cdots \xrightarrow{\delta} A[\xi|x]_{[n]} \xrightarrow{\delta} \cdots,$$

and $\mathfrak{C}_A$ is a N_0-graded A-module, where N_0 is the set of all non-negative integers.

Since A is a direct sum of R-modules, we may consider another decomposition of $\mathfrak{C}_A$. Let p be an integer with $p \leq t$. We shall construct a subcomplex C_p of C. Let $I = (i_1, \ldots, i_s)$ and $J = (j_1, \ldots, j_t)$ be sequences of non-negative integers, and we write $|I| = i_1 + \cdots + i_s$ and $|J| = j_1 + \cdots + j_t$. Put $\xi^I = \xi_1^{i_1} \cdots \xi_s^{i_s}$ and $x^J = x_1^{j_1} \cdots x_t^{j_t}$. Set

$$C_p^q = \bigoplus_{|I|+|J|=q} E_{p+q-|I|}\xi^I x^J.$$

Proposition 4.1. *Under the condition as above,*

$$C_p:\quad 0 \longrightarrow C_p^0 \xrightarrow{\delta} C_p^1 \xrightarrow{\delta} \cdots \xrightarrow{\delta} C_p^n \xrightarrow{\delta} \cdots.$$

is a subcomplex of C. C_p is a finite complex for each and we have that $C = \oplus_{p\leq t} C_p$.

Proof. Let $I = (i_1, \ldots, i_k, \ldots, i_s)$ and $J = (j_1, \ldots, j_k, \ldots, j_s)$. We put $I' = (i_1, \ldots, i_k + 1, \ldots, i_s)$ and $J' = (j_1, \ldots, j_k + 1, \ldots, j_s)$. Then it follows that

$$|I'| + |J| = |I| + |J'| = |I| + |J| + 1$$

$$r_k\xi_k E_{p+q-|I|}\xi^I x^J \subset E_{p+q+1-|I'|}\xi^{I'} x^J,$$

$$e_k x_k E_{p+q-|I|}\xi^I x^J \subset E_{p+q+1-|I|}\xi^I x^{J'}.$$

These relations show that $\delta C_p^q \subset C_p^{q+1}$, and therefore C_p is a subcomplex of C.

Next we prove that C_p is a finite complex. It is sufficient to see that it $q \geq t + s + 1 - p$, then $C_p^q = 0$. Assume $q \geq t + s + 1 - p$. Note that if $|I| < s + 1$, then $p + q - t - 1 \geq |I|$. Suppose $|I| + |J| = q$. If $|I| \geq s + 1$, then $\xi^I = 0$ and so $E_{p+q-|I|}\xi^I x^J = 0$. If $|I| < s + 1$, then $p + q - t - 1 \geq |I|$, and hence $p + q - |I| \geq t + 1$. Consequently $E_{p+q-|I|}\xi^I x^J = 0$, whence $C_p^q = 0$.

Finally we prove $C = \oplus_{p\leq t} C_p$. It suffices to show $A[\xi|x]_{[q]} = \oplus_{p\leq t} C_p^q$. If $|I| + |J| = q$, then $0 \leq q - |I|$, thus $t \leq t + q - |I|$. This implies $\oplus_{p\leq t} E_{p+q-|I|} = \oplus_{i\geq 0} E_i = A$. Therefore we get that

$$\bigoplus_{p\leq t} C_p^q = \bigoplus_{|I|+|J|=q} \bigoplus_{p\leq t} E_{p+q-|I|}\xi^I x^J = \bigoplus_{|I|+|J|=q} A\xi^I x^J = A[\xi|x]_{[q]}$$

We complete to prove our proposition.

We can express more precisely the length of the complex C_p.

Proposition 4.2. *Let the notation be as in proposition* 4.1 *and let p be an integer with $p \leq 0$. Then we have that*

$$C_p^q = 0 \quad \textit{for} \quad q < -p, \ t + s - p < q,$$
$$C_p^{-p} \neq 0 \quad \text{and} \quad C_p^{t+s-p} \neq 0.$$

Proof. In fact if $q < -p$, then $p + q - |I| < 0$ for all I. Thus $E_{p+q-|I|} = 0$ and it follows that $C_p^q = \oplus_{|I|+|J|=q} E_{p+q-|I|}\xi^I x^J = 0$. By the proof of proposition 4.1 if $t + s - p < q$, $C_p^q = 0$.

Now let $|I| = 0$ and $|J| = -p$. Then we obtain that $E_0 x^J \neq 0$ and so $C_p^{-p} = \oplus_{|I|+|J|=-p} E_{-|I|}\xi^I x^J \neq 0$. Next set $|I| = s$ and $|J| = t - p$. Then we see that $E_{t+s-|I|} = E_t \neq 0$, $\xi^I \neq 0$ and hence $E_{t+s-|I|}\xi^I x^J \neq 0$. This implies that $C_p^{t+s-p} = \oplus_{|I|+|J|=t+s-p} E_{t+s-|I|}\xi^I x^J \neq 0$.

References

[1] Y. I. Manin: "Gauge Field Theory and Complex Geometry", Springer-Verlag, 1988

[2] D. G. Northcott: "Multilinear algebra", Cambridge University Press, 1984

[3] M. Sakaguchi: "An Exact Sequence of Koszul Tranforms of Modules", *J. of Economic Sciences* No. **1**, Vol. **2** (1998), pp 1–8

[4] T. Schmitt: "Regular Sequences in Z_2-Graded Commutative Algebra", *J. of Algebra* **124** (1989), pp 60–118