The Grade of the Ideal Generated by Homogeneous Elements of Degee One in Associated Graded Rings

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Let R be a local ring with infinite residue field and I an ideal in R. Huckaba [2] showed that the grade of the ideal $gr_I(R)_+$ in the associated graded ring $gr_I(R)$ of R with respect to I plays an important role in deciding whether the reduction number of I is independent or not. In order to study the theory of reductions he investigated properties of the associated graded ring by making use of minimal reductions. In this paper we will develop this concept by using the results in Valabrega and Valla [9] which gives us much information about regular sequences in associated rings. The main result may be stated as follows. If $(x_1, ..., x_r)R$ is a minimal reduction of I with $\ell(I)=r$, then grade $gr_I(R)_+=\text{grade}\,(x_1^*, \ldots, x_r^*)gr_I(R)$ where x_i^* is the initial form of x_i in $gr_I(R)$.

Let $\bar{I}$ be the integral closure of I. We shall consider the inequality grade $gr_I(R)_+ \leq$ grade $gr_{\bar{I}}(R)_+$. This inequality does not hold in general. In the case the analytic spread of I is one, the author studied in [7] an ideal $\tilde{I}$ which is the smallest ideal with properties that $I \subseteq \tilde{I} \subseteq \bar{I}$ and grade $gr_{\tilde{I}}(R)_+ = 1$. We will search some sufficient conditions in this paper that the inequality holds.

For an ideal I in a noetherian ring R we denote by $gr_I(R)$ the associated graded ring $\oplus_{n=0}^{\infty} I^n / I^{n+1}$ of R with respect to I. If $x \in I^n$ and $x \notin I^{n+1}$, then we put $x^* = x + I^{n+1}$ in I^n / I^{n+1} and we call it the initial form of x. For $x \in \bigcap_{n=1}^{\infty} I^n$, put $x^* = 0$. Let J be an ideal in R. We shall denote by J^* the homogeneous ideal of $gr_I(R)$ generated by all the initial forms of the elements in J. For an ideal I, $\bar{I}$ denotes the integral closure of I. In [6] Northcott and Rees introduced and developed the concept of reductions. It is shown there that if (R, M) is a local ring with R/M infinite, then every ideal I contains a minimal reduction and any minimal reduction of I has a minimal basis consisting of $\ell(I)$ elements, where $\ell(I)$ is the analytic spread of the ideal I. It is well known that an ideal J with $J \subset I$ is a reduction of I if and only if $\bar{J} = \bar{I}$. An ideal I is said to be a regular ideal if grade $I > 0$.

Throughout this paper R will be denote a noetherian commutative ring with identity.

Let $R = \oplus_{n \geq 0} R_n$ be a graded ring and $M = \oplus_{n \geq 0} M_n$ a graded R-module. Then we denote the graded submodule $\oplus_{n \geq c} M_n$ of M by $M_{n \geq c}$ where c is an integer. In particular R_+ denotes $R_{\geq 1}$. Theorem 1.1 in Valabrega and Valla [9] suggests the following proposition (the case $c = 1$ in the following Proposition 1 is Theorem 1.1 in [9]). The virtually identical proof shall be omitted.

Proposition 1. *Let I and J be ideals in a ring R with $J \subset I$, and c be a positive integer. Suppose that $J = (x_1, \ldots, x_r)R$ and $x_i \notin I^2$ for every i. Then $J^*_{\geq c} = (x^*_1, \ldots, x^*_r) gr_I(R)_{\geq c}$ in $gr_I(R)$ if and only if $JI^{n-1} = J \cap I^n$ for all $n \geq c$.*

Corollary 2. *Let (R, M) be a local ring with infinite residue field, and let I be an ideal of R. Let J be a minimal reduction of I. Suppose $J = (x_1, \ldots, x_r)R$ where $r = \ell(I)$. Then there is an integer c such that $J^*_{\geq c} = (x^*_1, \ldots, x^*_r) gr_I(R)_{\geq c}$.*

Proof. By Lemma 3 of section 1 in [6], for every i, $x_i \notin MI$ and so $x_i \notin I^2$. Since J is a reduction of I, there exists an integer c such that $JI^{c-1} = I^c$. Then, for all $n \geq c$, we know $JI^{n-1} = I^n$ and hence $JI^{n-1} = J \cap I^n$. Therefore this corollary follows from Proposition 1.

Let I be an ideal in R and s be a positive integer. If $x \in I^s - I^{s+1}$, then we have a sequence

$$0 \longrightarrow gr_I(R) \xrightarrow{x^*} gr_I(R) \longrightarrow gr_{I/xR}(R/xR) \longrightarrow 0$$

where the x^* above the arrow identifies the map as multiplication by x^*. Since $x^* gr_I(R) \subseteq (xR)^*$, this is a complex.

Proposition 3. *Under the assumptions as above, let c be an integer. Then the sequence*

$$0 \longrightarrow gr_I(R)_{\geq c} \xrightarrow{x^*} gr_I(R)_{\geq c+s} \longrightarrow gr_{I/xR}(R/xR)_{\geq c+s} \longrightarrow 0$$

is exact if and only if the following conditions hold:

(i) $I^c \cap (I^{n+s+1} : x) = I^{n+1}$ *for all* $n \geq c$.

(ii) $xI^c = I^{c+s} \cap xR$.

Proof. The condition (i) means just that $0 \longrightarrow gr_I(R)_{\geq c} \xrightarrow{x^*} gr_I(R)_{\geq c+s}$ is exact.

Now we see that

$$\begin{aligned}&\mathrm{Im}\,[gr_I(R)_{\geq c} \xrightarrow{x^*} gr_I(R)_{\geq c+s}]\\ &\quad = (xI^c + I^{c+s+1})/I^{c+s+1} \oplus \cdots \oplus (xI^n + I^{n+s+1})/I^{n+s+1} \oplus \cdots\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Ker}\,[gr_I(R)_{\geq c+s} \longrightarrow gr_{I/xR}(R/xR)_{\geq c+s}] = (xR)^*_{\geq c+s}\\ &\quad = (xR \cap I^{c+s} + I^{c+s+1})/I^{c+s+1} \oplus \cdots \oplus (xR \cap I^{n+s} + I^{n+s+1})/I^{n+s+1} \oplus \cdots.\end{aligned}$$

Thus the sequence is exact at the middle term if and only if $xI^n + I^{n+s+1} = xR \cap I^{n+s} + I^{n+s+1}$ for all $n \geq c$. By the same method of the proof of Theorem 1.1 in [9], this statement is equivalent to the condition that $xI^n = xR \cap I^{n+s}$ for all $n \geq c$. Therefore we proved the part "only if".

Conversely to prove the part "if", it is sufficient to show that $xI^n = xR \cap I^{n+s}$ for all $n \geq c$, because the natural map $gr_I(R)_{\geq c+s} \longrightarrow gr_{I/xR}(R/xR)_{\geq c+s}$ is surjective in general. We use induction on n. By assumption (ii) we may suppose that $n > c$ and $xI^{n-1} = xR \cap I^{n+s-1}$. Assume $y \in xR \cap I^{n+s}$. Then $y \in xR \cap I^{n+s-1} = xI^{n-1}$, we can write $y = xr$ with $r \in I^{n-1}$. It follows from $n-1 \geq c$ and the condition (i) that $r \in (I^{n+s} : x) \cap I^c = I^n$. We therefore see $y \in xI^n$, so $xR \cap I^{n+s} \subset xI^n$. Hence $xR \cap I^{n+s} = xI^n$. This completes the proof.

Theorem 4. *Let R be a local ring with infinite residue field and I an ideal in R with $\ell(I) = r$. Let J be a minimal reduction of I. If $J = (x_1, \ldots, x_r)R$, then we have* grade $(x_1^*, \ldots, x_r^*)gr_I(R) = \text{grade}\; gr_I(R)_+$.

Proof. Since J is a reduction of I, we can find an integer m such that $JI^m = I^{m+1}$. Then for all $n \geq m$, $JI^n = I^{n+1}$ and hence $J \cap I^{n+1} = I^{n+1}$. Let $[J^*]_n$ be the set of homogeneous elements of degree n in J^* and put $c = m + 1$. Accordingly if $n \geq c$, then $[J^*]_n = (J \cap I^n + I^{n+1})/I^{n+1} = I^n/I^{n+1}$, and this impies $J^*_{\geq c} = gr_I(R)_{\geq c}$. By Corollary 2 we see that $(x_1^*, \ldots, x_r^*)gr_I(R)_{\geq c} = gr_I(R)_{\geq c}$.

Note that if P is a prime ideal in $gr_I(R)$, then $(x_1^*, \ldots, x_r^*)gr_I(R)_{\geq c} \subset P$ if and only if $(x_1^*, \ldots, x_r^*)gr_I(R) \subset P$. Thus it follows that $\sqrt{(x_1^*, \ldots, x_r^*)gr_I(R)} = \sqrt{(x_1^*, \ldots, x_r^*)gr_I(R)_{\geq c}}$. We also know $\sqrt{gr_I(R)_+} = \sqrt{gr_I(R)_{\geq c}}$, and so $\sqrt{(x_1^*, \ldots, x_r^*)gr_I(R)} = \sqrt{gr_I(R)_+}$. Therefore we obtain that grade $(x_1^*, \ldots, x_r^*)gr_I(R)$ = grade $gr_I(R)_+$, which completes the proof.

Now we consider the inequality

$$(*)\quad \text{grade}\; gr_I(R)_+ \leq \text{grade}\; gr_{\bar{I}}(R)_+$$

where I is an ideal in a local ring R. This inequality does not hold in general, and we shall study some sufficient conditions under which this holds.

Proposition 5. *Let R be a local ring and I an ideal in R. If* grade I = grade $gr_{\bar{I}}(R)_+$, *then* grade $gr_I(R)_+ \leq$ grade $gr_{\bar{I}}(R)_+$.

Proof. This is a direct consequence of the inequality grade $gr_I(R)_+ \leq$ grade I, which follows from (b) of Theorem 3.4 in [5].

By virtue of Proposition 3.1 in [9] and Theorem 4, if there is a minimal reduction J of $\bar{I}$ such that $J\bar{I} = \bar{I}^2$, then grade $gr_{\bar{I}}(R)_+ = \ell(I)$. Thus this is one of the sufficient conditions for that the inequality (*) holds. In particular it follows from Proposition 5.5 in [4] that if (R, M) is a regular local ring of dimension 2 with infinite residue field and I is an M-primary ideal, then the inequality (*) holds.

Let R be a local ring with infinite residue field and let I a regular ideal with $\ell(I) = 1$. Suppose that xR is a minimal reduction of I. Then x is a non-zero divisor. We denote by R' the integral closure of R in total quotient ring $Q(R)$ of R. Then $\bar{I} = xR' \cap R$ and $x \notin \bar{I}^2$. By Corollary 2.7 in [9], we see that x^* is a non-zero divisor in $gr_I(R)$ if and only if $x(\bar{I})^n = xR \cap (\bar{I})^{n+1}$ for all $n \geq 1$. Put $C_x = R :_{R'} x$ and we denote by $C_x^{[n]}$ the set consisting of all elements r such that r is a finite sum of elements $\alpha_{i_1} \cdots \alpha_{i_n}$ with $\alpha_{i_j} \in C_x$ for $1 \leq j \leq n$. Then $C_x^{[n]}$ is an R-module and $R \subseteq C_x \subseteq C_x^{[2]} \subseteq \cdots \subseteq C_x^{[n]} \subseteq \cdots \subseteq R'$.

Theorem 6. *Let R be a local ring with infinite residue field and let I a regular ideal with $\ell(I) = 1$. Supoose that xR is a minimal reduction of I. If $C_x^{[2]} = C_x^{[3]}$, then we have* grade $gr_{\bar{I}}(R)_+ = 1$.

Proof. Since xR is also a minimal reduction of $\bar{I}$, it follows from Theorem 4 that grade $gr_{\bar{I}}(R)_+ =$ grade $x^* gr_{\bar{I}}(R)_+$. Thus it is sufficient to prove that $x(\bar{I})^n = xR \cap (\bar{I})^{n+1}$ for all n ≥ 1. At first we shall show that $x\bar{I} = xR \cap \bar{I}^2$. Indeed, if $y \in xR \cap \bar{I}^2$, then $y = xr$ for some $r \in R$. Further y can be expressed in the form $y = x^2 \sum \alpha_i \alpha_j$ where $\alpha_i, \alpha_j \in C_x$, because $\bar{I}^2 = (xR' \cap R)^2$. Hence $r = x\sum \alpha_i \alpha_j$, which belongs to $xR \cap R'$, and so $y = xr \in x\bar{I}$.

Next Suppose $n \geq 2$ and we shall prove that $x(\bar{I})^n = xR \cap (\bar{I})^{n+1}$. If $y \in xR \cap (\bar{I})^{n+1}$, then we may express y in the form $y = xr = x^{n+1} \sum \alpha_{i_1} \cdots \alpha_{i_{n+1}}$, where $r \in R$ and $\alpha_{i_j} \in C_x$.

However we can write $\alpha_{i_1}\alpha_{i_2}\alpha_{i_3} = \sum \beta_{k_1}\beta_{k_2}$ for suitable elements β_{k_1}, β_{k_2} in C_x because of $C_x^{[2]} = C_x^{[3]}$. This yields $r = x^n \sum \beta_{k_1}\beta_{k_2}\alpha_{i_4}\cdots\alpha_{i_{n+1}}$, whence $r \in (xR' \cap R)^n$, and therefore $y \in x(\bar{I})^n$. The opposite inclusion is obvious and we complete the proof.

Remark 7. By the former half of the proof of Theorem 6, we see that if $x\bar{I}^2 = \bar{I}^3$, then grade $gr_{\bar{I}}(R)_+ = 1$. In fact, $x(\bar{I})^n = (\bar{I})^{n+1} = xR \cap (\bar{I})^{n+1}$ for all $n \geq 2$. In the case dim $R = 1$, this fact follows from Theorem 2.1 in [8]. Now we know that if x belongs to the conductor $R : R'$, then $C_x^{[2]} = C_x^{[3]} = R'$ because $C_x = R'$. On the other hand x^* is a non-zero divisor in the ring $gr_{\bar{I}}(R)$ if $\overline{x^n R} = \overline{xR}^n$ for all $n \geq 1$ since we know that $\overline{x^{n+1}R} : x = \overline{x^n R}$ for all $n \geq 1$ provided x is a non-zero divisor.

Corollary 8. *Let (R, M) be a local ring with R/M infinite and R' be the integral closure of R in its total quotient ring.* Suppose $MR' \subseteq R$. *If I is an ideal in R with $\ell(I) = 1$, then* grade $gr_I(R)_+ \leq$ grade $gr_{\bar{I}}(R)_+$.

Proof. Since $\ell(\mathrm{I}) = 1$, there exists an element x in R such that xR is a minimal reduction. Thus xR is also a minimal reduction of $\bar{I}$ because $\overline{xR} = \bar{I}$. By Theorem 4, grade $gr_I(R)_+ =$ grade $x^* gr_I(R)$ and grade $gr_{\bar{I}}(R)_+ =$ grade $x^* gr_{\bar{I}}(R)$. Therefore we know that if grade $I = 0$, then grade $gr_I(R)_+ = 0$ and that if grade $I = 1$, then grade $gr_{\bar{I}}(R)_+ = 1$ by virtue of Thoerem 6 and Remark 7. This completes the proof.

Proposition 9. *The following conditions are equivalent for a local ring R.*

(i) grade $gr_I(\mathrm{R})_+ \leq$ grade $gr_{\bar{I}}(R)_+$ *for all ideal with $\ell(I)$ = 1.*

(ii) grade I = grade $gr_{\bar{I}}(R)_+$ *for all ideals I with $\ell(I)$ = 1.*

Proof. (ii) $\Rightarrow$ (i) follows from Proposition 5.

(i) $\Rightarrow$ (ii): Put $R(X) = R[X]_{M[X]}$, where X is a trancendental element over R and M is the maximal ideal of R. Then we see that grade I = grade $IR(X)$ and the natural isomorphism $gr_{R(X)}(R(X)) \cong gr_I(R) \otimes_R R(X)$ shows that grade $gr_{IR(X)}(R(X))_+ =$ grade $gr_I(R)_+$ and $\ell(IR(X)) = \ell(I)$. Thererfore we may assume that the residue field of R is infinite. Let I be an ideal in R with $\ell(I) = 1$. Recall that there exists an element x in R such that grade $gr_{\bar{I}}(R)_+$ = grade $x^* gr_{\bar{I}}(R)$. If grade $I = 0$, then x is a zero divisor in R, whence x^* is a zero divisor in $gr_{\bar{I}}(R)$ and so grade $gr_{\bar{I}}(R)_+ = 0$. Now, if grade $I = 1$, then x is a non-zero divisor in R. Put $J = xR$.

Then $gr_J(R) \cong (R/xR)[X\}$ and $\bar{J} = \bar{I}$. It is clear that grade $gr_J(R)_+ = 1$. Thus we obtain grade $gr_{\bar{I}}(R)_+ = 1$ by the assunption and we complete the proof.

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