Analysis of the Optimum Ordering Quantity in Dynamic Inventory Models

Michinori Sakaguchi and Masanori Kodama

(Received on October 4, 2004)

Abstract

A mathematical model of the probabilistic inventory problems is presented and an analysis on the economic ordering quantity in N-period inventory problems is studied. The optimum policy of multistage problems is discussed under the assumption that the total cost function of single period follows some conditions. To seek the optimum policies we make functions and using them we are able to decide the economic ordering quantity. We give an example which follows our mathematical model.
Key words: Probabilistic inventory problem, Dynamic programming, Optimum policy.

Introduction

Mathematical inventory models with a piecewise cost function have been studied in Kodama and Sakaguchi (2001a, 2001b, 2001c, 2002), Sakaguchi and Kodama (2002a, 2002c) and various properties of an optimum policy in its inventory systems are obtained. It is also attempted to express the optimum function by closed forms with known cost functions and some sufficient conditions on cost functions are found to ensure simple treatments on an optimum policy. Our decision criterion is the minimization of expected costs which include the ordering, holding, and shortage costs.

In this paper we review an example in Sakaguchi and Kodama (2002b) which inspired us to make a generalization of the inventory model. Let x be the amount on hand before an order is placed and let z be the amount on hand in initial period after an order is received. In the probabilistic inventory problems of single period with demand B we would make decisions of ordering quatities to minimize the expectation $E\{C(B,z)\}$ of its total cost. We define the function $H(z)$ by the equation $E\{C(B,z)\} = -cx + H(z)$ in order to analyze the inventory system, where c is the purchasing cost per unit. The assumptions of the former models are as follows.

Let $R_1, \cdots, R_m$ be a sequence of real numbers such that $R_1 < \cdots < R_m$. Let

$H_i(z)\,(2 \le i \le m)$ be real valued-functions defined on $[R_{i-1},\ R_i]$, and $H_1(z)$ is defined on $(-\infty,\ R_1]$ and $H_{m+1}(z)$ is defined on $[R_m,\ \infty]$. Abbreviate $(-\infty,\ R_1]$ to $[R_0,\ R_1]$ and $[R_m,\ \infty)$ to $[R_m,\ R_{m+1}]$. Assume that for all i with $1 \le i \le m+1$, $H_i(z)$ has a continuous second derivative on $[R_{i-1},\ R_i]$, and it is a convex function on $[R_{i-1},\ R_i]$ which means by the condition that $H_i''(z) \ge 0$ on $(R_{i-1},\ R_i)$. We assume that $H_-'(R_i) \le H_+'(R_i)$ for all i, $\lim_{z\to-\infty} H_1'(z) < 0$ and $\lim_{z\to\infty} H_{m+1}'(z) > c$. Now we set

$$H(z) = H_i(z) \qquad \text{for} \quad z \in (R_{i-1},\ R_i].$$

The example in Sakaguchi and Kodama (2002b) which is simple does not satisfy these conditions in general. In this paper we take some of the assumptions such that the mathmatical model covers its example and we may still apply the developed theory. By the method of the dynamic programming we investigate the optimum ordering quantity of multiperiod problems provided we know about the function $H(z)$.

Let $f_1(x)$ be the minimal expectation of the total cost with single period. Then we may write

$$f_1(x) = \min_{z\ge x}\{-cx + H(z)\}.$$

Let $\phi(b)$ be the probability density function of demand B. In multiperiod models we suppose that $\phi(b)$ remains unchanged from period to period and demands in each period are independent. We should take in the discounted value of money in this case. That is, if $\alpha(<1)$ is the discount factor per period and $f_n(x)$ is the discount expected loss for n-period inventory model when an optimum policy is used at each purchaing opportunity, then

$$f_n(x) = \min_{z\ge x}\left\{ -cx + H(z) + \alpha\int_0^\infty f_{n-1}(z-b)\phi(b)db \right\}.$$

We present the fundamental analysis of our model in Theorem 1.6 and a method to get the optimum quantity in Theorem 1.7. An example which follows our model is shown in section 2.

1. Mathematical models

Let c, α be real numbers with $0 < c,\ 0 < \alpha < 1$. Let $\phi(b)$ be the density function of a real random variable B which means demand and we assume that $\phi(b)$ is a continuous function on $[0,\ \infty)$ with $\phi(b) = 0$ for $b < 0$. Let $\Phi(b)$ be its distribution function. We let $H(z)$ be a function on $\boldsymbol{R}$, which suggests that the expectation of the total cost in the probabilistic inventory problems of single period is $-cx + H(z)$.

Throughout this section we assume the following conditions on $H(z)$.

(i) $H(z)$ is a piecewise continuous function on $\boldsymbol{R}$ and $H(z)$ has a minimal value at $z = \bar{x}_1$. More precisely if $z < \bar{x}_1$ then $H(z) > H(\bar{x}_1)$ and if $\bar{x}_1 \leq z$ then $H(\bar{x}_1) \leq H(z)$.

(ii) Let $R_1, \cdots, R_m$ be a sequence of real numbers such that $R_1 < \cdots < R_m$ and $R_1 \leq \bar{x}_1 < R_2$. Let $H_i(z)$ $(1 \leq i \leq m-1)$ be real valued-functions defined on $[R_i,\ R_{i+1}]$, and let $H_m(z)$ be defined on $[R_m,\ \infty)$. We abbreviate $[R_m,\ \infty)$ to $[R_m,\ R_{m+1}]$. We suppose that $H_i(z)$ $(1 \leq i \leq m)$ has a continuous derivative on $[R_i,\ R_{i+1}]$, and we assume

$$H(z) = H_i(z) \quad \text{if} \quad z \in [R_i,\ R_{i+1}] \quad (1 \leq i \leq m),$$

which leads us that $H(z)$ is continuous on $[R_1,\ \infty)$.

(iii) $H'_+(z)$ is non-decreasing on $[\bar{x}_1,\ \infty)$.

(iv) We have $\lim_{z \to \infty} H'(z) > c$.

Note that $H'(z)$ is a piecewise continuous on $[R_1,\ \infty)$. For a given real number x and z we define functions $f_k(x)$, $F_k(z)$ $(k = 1, \cdots, N)$ as follows.

$$\begin{aligned} f_1(x) &= \min_{z \geq x}\{-cx + H(z)\}, \\ f_k(x) &= \min_{z \geq x}\left\{ -cx + H(z) + \alpha \int_0^\infty f_{k-1}(z-b)\phi(b)db \right\}, \quad k = 2, 3, \cdots, N, \end{aligned} \tag{1.1}$$

$$F_{k-1}(z) = H(z) + \alpha \int_0^\infty f_{k-1}(z-b)\phi(b)db, \quad f_0(\cdot) = 0, \quad k = 1, 2, \cdots, N. \tag{1.2}$$

As we say in introduction, the assumptions on $H(z)$ in this paper implies in essential the conditions in Kodama and Sakaguchi (2001a, 2001b, 2001c, 2002), Sakaguchi and Kodama (2002a, 2002c) and we could use the former result.

We consider the inventory problem of one-period. Since

$$f_1(x) = \begin{cases} -cx + H(\bar{x}_1) & \text{for} \quad x \leq \bar{x}_1, \\ -cx + H(x) & \text{for} \quad \bar{x}_1 \leq x, \end{cases} \tag{1.3}$$

we get the *optimum ordering quantity* in our inventory problem of single period that *if* $x \leq \bar{x}_1$*, then order* $\bar{x}_1 - x$*, otherwise do not order.*

Next we study 2-period problems. Since

$$F_1(z) = H(z) + \alpha \int_0^\infty f_1(z-b)\phi(b)db,$$

$$f_2(x) = \min_{z \geq x}\{-cx + H(z) + \alpha \int_0^\infty f_1(z-b)\phi(b)db\}$$
$$= \min_{z \geq x}\{-cx + F_1(z)\},$$

if we find a number $\bar{x}_2$ such that the function $F_1(z)$ has a minimal value at $\bar{x}_2$, then we make a decision that *if* $x \leq \bar{x}_2$*, then order* $\bar{x}_2 - x$*, otherwise do not order. Next if a demand in the first period is* b_1*, then we may make the decision of the second period as one period inventory problem with an initial stock* $\max\{\bar{x}_2,\ x\} - b_1$.

Thererfore we shall repeat the above method replacing $H(z)$ with $F_1(z)$ in order to obtain the optimum ordering quantity of a 2-period problem. Because $z - b \leq \bar{x}_1$ if and only if $z - \bar{x}_1 \leq b$, we have

$$f_1(z-b) = \begin{cases} -c(z-b) + H(\bar{x}_1) & \text{for} \quad z - \bar{x}_1 \leq b, \\ -c(z-b) + H(z-b) & \text{for} \quad b \leq z - \bar{x}_1. \end{cases} \tag{1.4}$$

Let m be the mean of $\phi(b)$.

Lemma 1.1. *$F_1(z)$ is piecewise continuous on* $\boldsymbol{R}$ and

$$F_1(z) = \begin{cases} H(z) - \alpha cz + \alpha H(\bar{x}_1) + \alpha cm & \text{for} \quad z \leq \bar{x}_1, \\ H(z) - \alpha cz + \alpha cm + \alpha \int_0^{z-\bar{x}_1} H(z-b)\phi(b)db & \\ \qquad + \alpha H(\bar{x}_1)(1 - \Phi(z - \bar{x}_1)) & \text{for} \quad \bar{x}_1 \leq z, \end{cases}$$

in particular $F_1(z)$ is continuous on $[R_1,\ \infty)$.

Proof. Since we assume that $\phi(b) = 0$ for $b < 0$, it follows from (1.4) that

$$F_1(z) = H(z) + \alpha \int_0^\infty [-c(z-b) + H(\bar{x}_1)]\phi(b)db$$
$$= H(z) - \alpha cz + \alpha H(\bar{x}_1) + \alpha cm \qquad \text{for} \quad z \leq \bar{x}_1,$$

$$F_1(z) = H(z) + \alpha \Big\{ \int_0^{z-\bar{x}_1} [-c(z-b) + H(z-b)]\phi(b)db$$
$$+ \int_{z-\bar{x}_1}^\infty [-c(z-b) + H(\bar{x}_1)]\phi(b)db \Big\}$$

$$= H(z) - \alpha cz + \alpha cm + \alpha \int_0^{z-\bar{x}_1} H(z-b)\phi(b)db$$
$$+ \alpha H(\bar{x}_1)(1 - \Phi(z - \bar{x}_1)) \quad \text{for} \quad \bar{x}_1 \le z.$$

Since we assumed that $\phi(b)$ is a continuous function on $[0, \infty)$, we complete the proof.

By lemma 1.1 we have a derivative of $F_1(z)$ (cf. Theorem 2.6 in Sakaguchi and Kodama (2002a)).

Lemma 1.2. *We have*

$$F'_{1+}(R_1) = H'_+(R_1) - \alpha c,$$
$$F'_1(z) = H'(z) - \alpha c \qquad \text{for} \quad R_1 < z < \bar{x}_1,$$
$$F'_1(z) = H'(z) - \alpha c + \alpha \int_0^{z-\bar{x}_1} H'(z-b)\phi(b)db$$
$$\text{for} \quad z \ge \bar{x}_1 \quad \text{and} \quad z \ne R_i \quad (2 \le i \le m),$$
$$F'_{1+}(R_i) = H'_+(R_i) - \alpha c + \alpha \int_0^{R_i - \bar{x}_1} H'(R_i - b)\phi(b)db \quad (2 \le i \le m),$$
$$F'_{1-}(R_i) = H'_-(R_i) - \alpha c + \alpha \int_0^{R_i - \bar{x}_1} H'(R_i - b)\phi(b)db \quad (2 \le i \le m).$$

We see that $F'_1(z)$ exists on $[R_1, \infty)$ without finite points and $F'_1(z)$ is piecewise continuous on $[R_1, \infty)$.

Lemma 1.3. *We have* $\lim_{z\to\infty} F'_1(z) > c$.

Proof. It follows that

$$\lim_{z\to\infty} F'_1(z) = \lim_{z\to\infty} H'(z) - \alpha c + \alpha \int_0^{\infty} \lim_{z\to\infty} H'(z)\phi(b)db$$
$$= \lim_{z\to\infty} H'(z)(1+\alpha) - \alpha c > c,$$

and we finish the proof.

Note that if $\bar{x}_1 \ne R_1$, then $F_1(\bar{x}_1) = -\alpha c < 0$ and if $\bar{x}_1 = R_1$, then $F_{1+}(\bar{x}_1) = H'_{1+}(R_1) - \alpha c$.

Lemma 1.4. *Put* $\bar{x}_2 = \inf\{z \ge \bar{x}_1 \mid F'_{1+}(z) \ge 0\}$. *Then we have* $F_1(z) > F_1(\bar{x}_2)$ *for* $z < \bar{x}_2$ *and* $F_1(z) \le F_1(\bar{x}_2)$ *for* $\bar{x}_2 < z$.

PROOF. It follows from Lemma 1.3 that there exists a number $\bar{x}_2$ such that $\bar{x}_2 = \inf\{z \geq \bar{x}_1 \mid F'_{1+}(z) \geq 0\}$. We see that if $z < \bar{x}_1$, then $F_1(z) > F_1(\bar{x}_1)$. In fact,

$$\begin{aligned} F_1(z) - F_1(\bar{x}_1) &= \{H(z) + \alpha(H(\bar{x}_1) - cz) + \alpha cm\} \\ &\quad - \{H(\bar{x}_1) + \alpha(H(\bar{x}_1) - c\bar{x}_1) + \alpha cm\} \\ &= (H(z) - H(\bar{x}_1)) + \alpha c(\bar{x}_1 - z) > 0. \end{aligned} \tag{1.5}$$

If $\bar{x}_1 \leq z < \bar{x}_2$ and $z \neq R_i$ for any i, then $F'_1(z) < 0$. Since $F_1(z)$ is a continuous function on $[R_1, \infty)$, we see $F_1(z) > F(\bar{x}_2)$ for $\bar{x}_1 \leq z < \bar{x}_2$ by elementary facts.

If $\bar{x}_2 < z$, then $F'_1(z) \geq 0$. Indeed, it follows from Lemma 1.2 that

$$F'_{1+}(\bar{x}_2) = H'_+(\bar{x}_2) - \alpha c + \alpha \int_0^{\bar{x}_2 - \bar{x}_1} H'(\bar{x}_2 - b)\phi(b)db \geq 0.$$

Therefore we obtain by Lemma 1.2

$$\begin{aligned} F'_{1+}(z) &= H'_+(z) - \alpha c + \alpha \int_0^{z-\bar{x}_1} H'(z-b)\phi(b)db \\ &= H'_+(z) - \alpha c + \alpha \int_0^{\bar{x}_2-\bar{x}_1} H'(z-b)\phi(b)db + \alpha \int_{\bar{x}_2-\bar{x}_1}^{z-\bar{x}_1} H'(z-b)\phi(b)db \\ &\geq H'_+(z) - H'_+(\bar{x}_2) + \alpha \int_0^{\bar{x}_2-\bar{x}_1} [H'(z-b) - H'(\bar{x}_2-b)]\phi(b)db \\ &\qquad + \alpha \int_{\bar{x}_2-\bar{x}_1}^{z-\bar{x}_1} H'(z-b)\phi(b)db. \end{aligned} \tag{1.6}$$

Since $H'_+(z)$ is non-decresing on $[\bar{x}_1, \infty)$, we see that

$$\begin{aligned} &H'_+(z) - H'_+(\bar{x}_2) \geq 0, \\ &H'(z-b) - H'(\bar{x}_2 - b) \geq 0 \qquad \text{for} \quad 0 \leq b \leq \bar{x}_2 - \bar{x}_1, \\ &\qquad\qquad z - b \neq R_i,\ \bar{x}_2 - b \neq R_i \quad (2 \leq i \leq m), \\ &H'(z-b) \geq H'_+(\bar{x}_1) \geq 0 \quad \text{for} \quad \bar{x}_2 - \bar{x}_1 \leq b \leq z - \bar{x}_1, \\ &\qquad\qquad z - b \neq R_i \quad (2 \leq i \leq m). \end{aligned} \tag{1.7}$$

Thus we get $F'_1(z) \geq 0$ for $\bar{x}_2 < z$ and $z \neq R_i$ $(2 \leq i \leq m)$ by (1.6) and (1.7). Hence it is also clear that $F_1(\bar{x}_2) \leq F_1(z)$ for $\bar{x}_2 \leq z$. We conclude the proof of the lemma.

Lemma 1.5. *We have*

$$f_2(x) = \begin{cases} -cx + F_1(\bar{x}_2), & \text{for} \quad x \leq \bar{x}_2, \\ -cx + F_1(x), & \text{for} \quad \bar{x}_2 \leq x. \end{cases}$$

PROOF. By (1.1) and (1.2) we see that

$$f_2(x) = \min_{z \geq x} \left\{ -cx + H(z) + \alpha \int_0^\infty f_1(z-b)\phi(b)db \right\} = \min_{z \geq x} \{-cx + F_1(z)\}.$$

Therefore we prove this lemma by Lemma 1.4.

Lemma 1.6. *Put* $F_{1i}(z) = F_1(z)$ *for* $z \in [R_i,\ R_{i+1}]$ $(1 \leq i \leq m)$. *Then* $F_{1i}(z)$ *has a derivative on* $[R_i,\ R_{i+1}]$.

Lemma 1.7. $F'_{1+}(z)$ *is non-decreaing on* $[\bar{x}_2,\ \infty)$.

Proof. Assume $\bar{x}_2 \leq z_1 < z_2$. By Lemma 1.2 we see

$$\begin{aligned} F'_{1+}(z_2) - F'_{1+}(z_1) &= H'_+(z_2) - H'_+(z_1) \\ &\quad + \alpha \int_0^{z_2-\bar{x}_1} H'(z_2-b)\phi(b)db - \alpha \int_0^{z_1-\bar{x}_1} H'(z_1-b)\phi(b)db \\ &= H'_+(z_2) - H'_+(z_1) + \alpha \int_0^{z_1-\bar{x}_1} [H'(z_2-b) - H'(z_1-b)]\phi(b)db \\ &\quad + \alpha \int_{z_1-\bar{x}_1}^{z_2-\bar{x}_1} H'(z_2-b)\phi(b)db \geq 0. \end{aligned}$$

Because $H'_+(z)$ is non-decreasing on $[\bar{x}_1,\ \infty)$,

$$\begin{aligned} &H'_+(z_2) - H'_+(z_1) \geq 0, \\ &H'(z_2-b) - H'(z_1-b) \geq 0 \quad \text{for} \quad 0 \leq b \leq z_1 - \bar{x}_1, \\ &\qquad\qquad z_2 - b \neq R_i,\ z_1 - b \neq R_i\ (2 \leq i \leq m), \\ &H'(z_2-b) \geq H'_+(\bar{x}_1) \geq 0 \quad \text{for} \quad z_1 - \bar{x}_1 \leq b \leq z_2 - \bar{x}_1, \\ &\qquad\qquad z_2 - b \neq R_i\ (2 \leq i \leq m). \end{aligned}$$

We complete the proof.

By induction we get the following theorem because we have assumed $F_0(z) = H(z)$.

Theorem 1.8. *For each* $i\ (1 \leq i \leq N)$ *we have the following statements.*

(1) *We have*

$$F_i(z) = \begin{cases} H(z) - \alpha cz + \alpha F_{i-1}(\bar{x}_i) + \alpha cm & \text{for} \quad z \leq \bar{x}_i, \\ H(z) - \alpha cz + \alpha cm + \alpha \displaystyle\int_0^{z-\bar{x}_i} F_{i-1}(z-b)\phi(b)db & \\ \qquad\qquad + \alpha F_{i-1}(\bar{x}_i)(1 - \Phi(z - \bar{x}_i)) & \text{for} \quad \bar{x}_i \leq z, \end{cases}$$

in particular $F_i(z)$ *is continuous on* $[R_1, \infty)$.

(ii) *We obtain that*

$$F'_{i+}(R_j) = H'_{+}(R_j) - \alpha c, \quad \text{for} \quad R_j \le \bar{x}_i,$$
$$F'_{i-}(R_j) = H'_{-}(R_j) - \alpha c, \quad \text{for} \quad R_1 < R_j \le \bar{x}_i,$$
$$F'_i(z) = H'(z) - \alpha c \qquad \text{for} \quad R_1 < z < \bar{x}_i,$$
$$F'_i(z) = H'(z) - \alpha c + \alpha \int_0^{z-\bar{x}_i} F'_{i-1}(z-b)\phi(b)db$$
$$\text{for} \quad z \ge \bar{x}_i \quad \text{and} \quad z \ne R_j \quad (\bar{x}_i \le R_j),$$
$$F'_{i+}(R_j) = H'_{+}(R_j) - \alpha c + \alpha \int_0^{R_j-\bar{x}_i} F'_{i-1}(R_j-b)\phi(b)db \quad \text{for} \quad \bar{x}_i \le R_j,$$
$$F'_{i-}(R_j) = H'_{-}(R_j) - \alpha c + \alpha \int_0^{R_j-\bar{x}_i} F'_{i-1}(R_j-b)\phi(b)db \quad \text{for} \quad \bar{x}_i \le R_j.$$

(iii) *We have* $\lim_{z\to\infty} F'_i(z) > c$.

(iv) *There is a number* $\bar{x}_{i+1}$ *such that* $\bar{x}_{i+1} = \inf\{z \ge \bar{x}_i \mid F'_{i+}(z) \ge 0\}$.

(v) $F_i(z)$ *is a piecewise continuous function on* $\boldsymbol{R}$ *and* $F_i(z)$ *has a minimal value at* $z = \bar{x}_{i+1}$. *More precisely if* $z < \bar{x}_{i+1}$ *then* $F_i(z) > F_i(\bar{x}_{i+1})$ *and if* $\bar{x}_{i+1} < z$ *then* $F_i(\bar{x}_i) \le F_i(z)$.

(vi) *We obtain that*

$$f_{i+1}(x) = \begin{cases} -cx + F_i(\bar{x}_{i+1}), & \text{for} \quad x \le \bar{x}_{i+1}, \\ -cx + F_i(x), & \text{for} \quad \bar{x}_{i+1} \le x. \end{cases}$$

(vii) *Put* $F_{ij}(z) = F_i(z)$ *for* $z \in [R_j, R_{j+1}]$ $(1 \le j \le m)$. *Then* $F_{ij}(z)$ *has a derivative on* $[R_j, R_{j+1}]$.

(viii) $F'_{i+}(z)$ *is a non-decreasing function on* $[\bar{x}_{i+1}, \infty)$.

Proof. It follows from lemmas above that this theorem holds in the case $i = 1$. If $i = 2$, then we see that the assumptions on $H(z)$ lead us to the ones on $F_1(z)$. Thus this theorem is true when $i = 2$. By the induction we may prove the theorem.

Theorem 1.9. *In our mathematical model of dynamic inventory problems the optimum ordering quantity of N-period is if the initial stock x is less than* $\bar{x}_N$, *then order* $x - \bar{x}_N$ *and otherwise do not order.*

Proof. By (v) in Theorem 1.8 we may make decisions in this theorem and we finish the proof.

We may apply the following analysis to this inventory model and we study the optimum policies by the results of Sakaguchi and Kodama (2002c).

Theorem 1.10 (Theorem 2.1 in Sakaguchi and Kodama (2002c)). *Let R be a real number. Then the following statements hold:*

(1) *If $H'_+(R) \geq \alpha c$, then $F'_{k+}(R) \geq 0$ for all k $1 \leq k \leq N-1$.*

(2) *If $1 \leq k \leq N-1$ and $F'_{(k-1)+}(R) \leq 0$, then $F'_{k+}(R) = H'_+(R) - \alpha c$.*

(3) *If $1 \leq k \leq N-1$, $H'_+(R) < \alpha c$ and $F'_{(k-1)+}(R) \leq 0$, then $F'_{k+}(R) < 0$.*

(4) *If $H'_+(R) < 0$, then $F'_{k+}(R) < 0$ for all k $1 \leq k \leq N-1$.*

(5) *If $1 \leq k \leq N-1$ and $H'_+(R) < \dfrac{\alpha c(1+\alpha+\cdots+\alpha^{k-1})}{1+\alpha+\cdots+\alpha^k}$, then $F'_{k+}(R) < 0$.*

(6) *If $1 \leq k \leq N-1$ and $H'_+(R) = \dfrac{\alpha c(1+\alpha+\cdots+\alpha^{k-1})}{1+\alpha+\cdots+\alpha^k}$, then $F'_{k+}(R) \leq 0$.*

Corollary. *Let p be an integer with $1 \leq p \leq m$. Then we have:*

(1) *If $H'_+(R_p) \geq \alpha c$, then $\bar{x}_k \leq R_p$ for all k $1 \leq k \leq N$.*

(2) *If $H'_+(R_p) < 0$, then $R_p < \bar{x}_k$ for all k $1 \leq k \leq N$.*

(3) *If $H'_+(R_p) < \dfrac{\alpha c(1+\alpha+\cdots+\alpha^{l-1})}{1+\alpha+\cdots+\alpha^l}$, then $R_p < \bar{x}_k$ for all k $l+1 \leq k \leq N$.*

2. An example

We shall consider an example of the probabilistic multi-period inventory model with zero delivery lag, backlogging of demand and linear purchasing cost $[c(y) = cy]$ in this section.

Model and Notations:

(1) The multi-period model with backlogging of demand will be investigated under general demand without setup cost. The stock replenishment occurs instantaneously.

(2) Regular ordering takes at the beginning of each period, purchasing cost c_1 is charged and the period length is t. Let x be the initial stock level and let z be the amount on hand in initial period after an order is received. That means that the amount of a regular order is $z - x$.

(3) Let h and p be the holding and shortage costs per unit per period, respectively. We assume $c_1 < p$.

(4) Demand B in each period is a nonnegative random variable with a known distribution $\Phi(b)$ and its density $\phi(b)$. The functions $\Phi(b)$ and $\phi(b)$ remain unchanged from period to period and demands in each period are independent.

(5) The additional orders in the case the amount in inventory is less than R at given t_0 $(0 < t_0 < t)$ in each period are allowed, and the stock is replenished to the amount in inventory at t_0 is equal to S. Let c_2 $(c_1 \leq c_2$)be the cost per unit of the additional orders, respectively.

(6) Demand occurs according to a general function $g(T/t)b$ during the period t. Let the amount in inventory at time T be designated by $Q(T)$. Then

$$Q(T) = z - g(T/t)b, \qquad (0 \leq T \leq t), \tag{2.1}$$

where $g(0) = 0, g(1) = 1$ and $dg(x)/dx > 0$ $(0 \leq x \leq 1)$. In other words, demand occurs according to $g(T/t)b$, $(0 \leq T \leq t)$. If $b \geq s$, then there exists an unique positive T/t such that $s = g(T/t)b$. Let it be designated by $T/t = g^{-1}(s/b)$.

(7) The total cost is the sum of the purchasing cost, the holding cost, the shortage cost and the additional purchasing cost and we search the amount of a regular order at which the expectation of the sum is minimal.

(8) We denote α by the discount factor $(0 < \alpha < 1)$. Let $f_n(x)$ be the discount expected loss for n-period inventory model provided that an optimal policy is used at each purchaing opportunity, where x is the initial stock level.

Since the unifilled demand is backlogged, it is necessary to investigate in the case when z is negative.

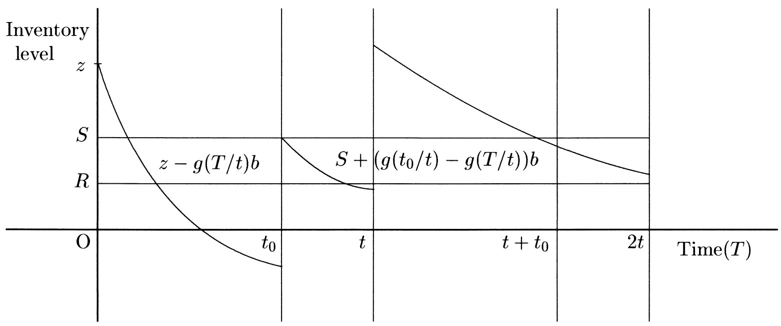

Figure 1

Let $T_0 = \dfrac{t_0}{t}$, $g_0 = g(T_0)$. Then $0 < T_0 < 1, 0 < g_0 < 1$. For the sake of simplicity we add the conditions that $g_0 \geq \dfrac{1}{2}$ and $R \leq g_0 S$ and we assume that $\phi(b)$ has a derivative on $(0, \infty)$.

Let's consider the sequence 0, R, $\dfrac{R}{1-g_0}$, $R+\dfrac{g_0S}{1-g_0}$. We set that

$$G(y)=\int_0^y g(T)dT \quad and \quad m=E(B).$$

Put the constants $d_i\,(1\le i\le 4)$ as follows.

$$\begin{aligned}
d_1&=\int_{t_0}^{\frac{S}{1-g_0}}[(1-T_0)(S+g_0b)-(G(1)-G(T_0))]\phi(b)db\\
d_2&=\int_{0}^{\frac{S}{1-g_0}}[(1-T_0)(S+g_0b)-(G(1)-G(T_0))]\phi(b)db\\
d_3&=\int_{\frac{S}{1-g_0}}^{\infty}\big[\big(g^{-1}((S/b)+g_0)-T_0\big)(S+g_0b)\\
&\qquad\qquad+\big(G(T_0)-G\big(g^{-1}((S/b)+g_0)\big)\big)b\big]\phi(b)db\\
d_4&=\int_{\frac{S}{1-g_0}}^{\infty}\big[\big(g^{-1}((S/b)+g_0)-1\big)(S+g_0b)\\
&\qquad\qquad+\big(G(1)-G\big(g^{-1}((S/b)+g_0)\big)\big)b\big]\phi(b)db
\end{aligned}\tag{2.2}$$

We let the functions $w_i\;(1\le i\le 11)$ be

$$\begin{aligned}
w_1(z)&=\int_0^{\infty}[G(T_0)b-T_0z]\phi(b)db,\\
w_2(z)&=\int_0^{\frac{z}{g_0}}[T_0z-G(T_0)b]\phi(b)db,\\
w_3(z)&=\int_{\frac{z}{g_0}}^{\infty}[G(T_0)b-T_0z]\phi(b)db,\\
w_4(z)&=\int_{\frac{z}{g_0}}^{\infty}\big[zg^{-1}(z/b)-G\big(g^{-1}(z/b)\big)b\big]\phi(b)db,\\
w_5(z)&=\int_0^{z}[(1-T_0)z-(G(1)-G(T_0))b]\phi(b)db,\\
w_6(z)&=\int_0^{\frac{z-R}{g_0}}[(1-T_0)z-(G(1)-G(T_0))b]\phi(b)db,\\
w_7(z)&=\int_{\frac{z-R}{g_0}}^{\frac{S}{1-g_0}}[(1-T_0)(S+g_0b)-(G(1)-G(T_0))b]\phi(b)db,\\
w_8(z)&=\int_{z}^{\frac{z-R}{g_0}}\big[\big(g^{-1}(z/b)-T_0\big)z+\big(G(T_0)-G(g^{-1}(z/b)\big)b\big]\phi(b)db,\\
w_9(z)&=\int_{z}^{\frac{z-R}{g_0}}\big[\big(g^{-1}(z/b)-1\big)z+\big(G(1)-G(g^{-1}(z/b)\big)b\big]\phi(b)db,\\
w_{10}(z)&=S+g_0m-z,\\
w_{11}(z)&=\int_{\frac{z-R}{g_0}}^{\infty}[S+g_0b-z]\phi(b)db
\end{aligned}\tag{2.3}$$

We also set the functions $H_i(z)$ $(1 \le i \le 5\ )$ as follows.

$$
\begin{aligned}
H_1(z) &= c_1 z + p w_1(z) + c_2 w_{10}(z) + h(d_1 + d_3) + p d_4, \qquad \text{for} \quad z \le 0, \\
H_2(z) &= c_1 z + h w_2(z) + (h+p) w_4(z) + p w_3(z) + c_2 w_{10}(z) \\
&\qquad + h(d_2 + d_3) + p d_4, \qquad \text{for} \quad 0 \le z \le R, \\
H_3(z) &= c_1 z + h(w_2(z) + w_6(z) + w_7(z)) + (h+p) w_4(z) + p w_3(z) \\
&\qquad + c_2 w_{11}(z) + h d_3 + p d_4, \qquad \text{for} \quad R \le z \le \frac{R}{1-g_0}, \\
H_4(z) &= c_1 z + h(w_2(z) + w_5(z) + w_7(z) + w_8(z)) + (h+p) w_4(z) \\
&\quad + p(w_3(z) + w_9(z)) + c_2 w_{11}(z) + h d_3 + p d_4, \ \text{for} \ \frac{R}{1-g_0} \le z \le R + \frac{g_0 S}{1-g_0}, \\
H_5(z) &= c_1 z + h(w_2(z) + w_5(z) + w_8(z)) + (h+p) w_4(z) + p(w_3(z) + w_9(z)) \\
&\qquad + c_2 w_{11}(z) + h d_3 + p d_4, \quad \text{for} \quad R + \frac{g_0 S}{1-g_0} \le z.
\end{aligned}
\tag{2.4}
$$

Then $H_i(z)$ $(1 \le i \le 5)$ have a second derivative on the suitable interval. The expected cost of single period is given by

$$
\begin{aligned}
E\{C(B,z)\} = c_1(z-x) &+ hE\{\text{holding quantity}\} + pE\{\text{shortage quantity}\} \\
&+ c_2 E\{\text{additional purchasing quantity}\}.
\end{aligned}
$$

We define the function $H(z)$ by the equation $E\{C(B,z)\} = -c_1 x + H(z)$. Then we have the following proposition.

Proposition 2.1. (Sakaguchi and Kodama (2002b)) *We have*

$$
H(z) = \begin{cases}
H_1(z) & \text{if} \quad z \le 0\,, \\
H_2(z) & \text{if} \quad 0 \le z \le R\,, \\
H_3(z) & \text{if} \quad R \le z \le \dfrac{R}{1-g_0}\,, \\
H_4(z) & \text{if} \quad \dfrac{R}{1-g_0} \le z \le R + \dfrac{g_0 S}{1-g_0}\,, \\
H_5(z) & \text{if} \quad R + \dfrac{g_0 S}{1-g_0} \le z\,.
\end{cases}
\tag{2.5}
$$

Now we shall show an example which needs our model in section 1. For that sake we set the density function $\phi(b)$ as follows.

$$
\phi(b) = \begin{cases}
0 & \text{if} \quad b < 0\,, \\
\dfrac{3(1-g_0)^3}{R^3}\left(b - \dfrac{R}{1-g_0}\right)^2 & \text{if} \quad 0 \le b < \dfrac{R}{1-g_0}\,, \\
0 & \text{if} \quad \dfrac{R}{1-g_0} \le b\,.
\end{cases}
\tag{2.6}
$$

Then we see that $w_7(z)=0$, $w_{11}(z)=0$ by (2.3), and hence $H_4(z)=H_5(z)$ by (2.4). We reset the function $H(z)$.

Proposition 2.2. *We obtain that*

$$H(z)=\begin{cases} H_1(z) & \text{if} \quad z\le 0\,, \\ H_2(z) & \text{if} \quad 0\le z\le R\,, \\ H_3(z) & \text{if} \quad R\le z\le \dfrac{R}{1-g_0}\,, \\ H_4(z) & \text{if} \quad \dfrac{R}{1-g_0}\le z. \end{cases} \tag{2.7}$$

The derivarives of $H_i(z)$ were shown in Sakaguchi and Kodama (2002b) when $\phi(b)$ is in general. We obtain the derivatives in this case by these facts.

Proposition 2.3. *Under the assumption 2.6 we have*

$$H_1'(z)=c_1-pT_0-c_2 \quad \text{if} \quad z\le 0,$$

$$H_2(z)=c_1+(h+p)\left[T_0\Phi\left(\frac{z}{g_0}\right)+\int_{\frac{z}{g_0}}^{\infty} g^{-1}\left(\frac{z}{b}\right)\phi(b)db\right]-pT_0-c_2 \quad \text{if} \quad 0\le z\le R,$$

$$\begin{aligned} H_3'(z)=c_1+h&\left[(1-T_0)\Phi\left(\frac{z-R}{g_0}\right)+T_0\Phi\left(\frac{z}{g_0}\right)+\int_{\frac{z}{g_0}}^{\infty} g^{-1}\left(\frac{z}{b}\right)\phi(b)db\right.\\ &\left.-\frac{1}{g_0}(1-T_0)(S-R)\phi\left(\frac{z-R}{g_0}\right)\right]+p\left[\int_{\frac{z}{g_0}}^{\infty} g^{-1}\left(\frac{z}{b}\right)\phi(b)db-T_0+T_0\Phi\left(\frac{z}{g_0}\right)\right]\\ &+c_2\left[\Phi\left(\frac{z-R}{g_0}\right)-1-\frac{S-R}{g_0}\phi\left(\frac{z-R}{g_0}\right)\right], \quad \text{if} \quad R\le z\le \frac{R}{1-g_0}, \end{aligned}$$

$$H_4'(z)=c_1+h \quad \text{if} \quad \frac{R}{1-g_0}\le z.$$

By Proposition 2.3 we obtain the following proposition.

Proposition 2.4. *Let $\phi(b)$ be as (2.6). Then we have the following statements,*

(i) *we have* $H_3'(R)=H_2'(R)-\left[\dfrac{h}{g_0}(1-T_0)(S-R)+\dfrac{c_2(S-R)}{g_0}\right]\phi(0)$, *hence* $H_2'(R)>H_3'(R)$.

(ii) *It is able to be $H_2'(R)<0$ for a suitable c_2.*

(iii) $H_3'\left(\dfrac{R}{1-g_0}\right)=H_4'\left(\dfrac{R}{1-g_0}\right)$.

(iv) $H_i''(z)\ge 0$ *for all* i $(1\le i\le 4)$.

(V) $\lim_{z \to \infty} H'(z) = c_1 + h$.

Now we give an example of the mathematical inventory model in section 1 if $H_2'(R) < 0$ and we put $\bar{x}_1 = \inf\{z \mid H_3'(z) \geq 0\}$.

The research of the paper we generously supported by a Grant from Sogo Kenkyu-sho (Institute of Advanced Studies), Hiroshima Shudo University in 2001–2003.

References

Kodama and Sakaguchi. (2001a). "Some Comments on the Probabilistic Dynamic Inventory Problems with Piecewise Cost Functions", Recent Dvelopments in Operations Research, Narosa, New Delhi, 191–196.

Kodama and Sakaguchi. (2001b). "Probabilistic analysis of dynamic inventory models with multiple piecewise cost functions", Journal of Information & Optimization Sciences 22, 57–72.

Kodama and Sakaguchi. (2001c). "Dynamic programming on the probabilistic inventory models with multiple piecewise cost functions", Journal of Information & Optimization Sciences 22, 419–440.

Kodama and Sakaguchi. (2002). "A Study on the Dynamic Inventory Models with Piecewise Cost Functions" (in Japanese), Monograph series 119, Institute for Advanced Studies, Hiroshima Shudo University.

Sakaguchi and Kodama.(2002a). "On the dynamic probabilistic inventory problems with piece-wise cost functions which may not be piecewise smooth", Bulletin of Informatics and Cyber-netics 34, Kyushu University, 75–90.

Sakaguchi and Kodama. (2002b). "A Remark on the Dynamic Inventory Models" (in Japanese), Journal of Economic Sciences 6, Hiroshima Shudo University, 41–70.

Sakaguchi and Kodama (2002c). "An application of dynamic probabilistic inventory models with piecewise cost functions", Journal of Information & Optimization Sciences, 23, No. 3, 523–549.